Numerical Study of Nonlinear R-modes in Neutron Stars

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What is r-mode?

- The r-modes of rotating neutron stars are some particular oscillation modes whose restoring force is the Coriolis force.

- In linear theory ($\alpha, \Omega \to 0$), r-modes are primarily mass-current perturbations:

$$\begin{aligned} \delta\vec{v} &= \alpha R \Omega \left(\frac{r}{R}\right)^l \vec{r} \times \nabla Y_{ll} e^{i\omega t} + O(\Omega^3) \\ \delta\rho &= O(\Omega^2) \end{aligned}$$

- When $\alpha \sim O(1)$, $E_{mode} \sim O(E_{rot})$

Why study r-mode?

- All r-modes are unstable due to GR reaction

  (Andersson 1998, Friedman & Morsink 1998)

- CFS mechanism

  (Chandrasekhar 1970, Friedman & Schutz 1978)

All rotating perfect-fluid stars are formally unstable to non-axisymmetric perturbations

The CFS mechanism

- GR carry angular momentum away from non-axisymmetric modes to ∞
- Non-rotating star:

  GR carry positive J away from a forward-moving (with $J > 0$) mode

  GR carry negative J away from a backward-moving (with $J < 0$) mode

All modes are damped in a non-rotating star

- Rotating star:

  A backward-moving (with $J < 0$) mode can be dragged forward relative to ∞. GR carry positive J away.

The amplitude of the mode must grow!

The CFS mechanism

- Condition for the CFS mechanism: $\omega_i \omega_r < 0$
- ω_i is the inertial frame angular frequency of the mode.
- ω_r is the rotating frame angular frequency of the mode.
- For the r-modes

$$\omega_i \omega_r = -\frac{2(l-1)(l+2)}{l^2(l+1)^2} m^2 \Omega^2 < 0$$

- All r-modes are unstable for arbitrarily small spin rate!

The r-mode instability

- The fastest growing ($l = m = 2$) r-mode has a growth time ≈ 40 sec for a “typical” neutron star model.
- 40 sec $\ll$ viscous-damping timescale (for 10^8 K $< T <$ 10^{10} K).
- As the mode grows....GR carries angular momentum away from the star....the star spins down rapidly.
- The growth of the mode will stop.......but by what?
- The instability expected in:

1. Hot, young rapidly rotating neutron stars.
2. Old neutron stars spun up by accretion in LMXBs.

Astrophysical implications

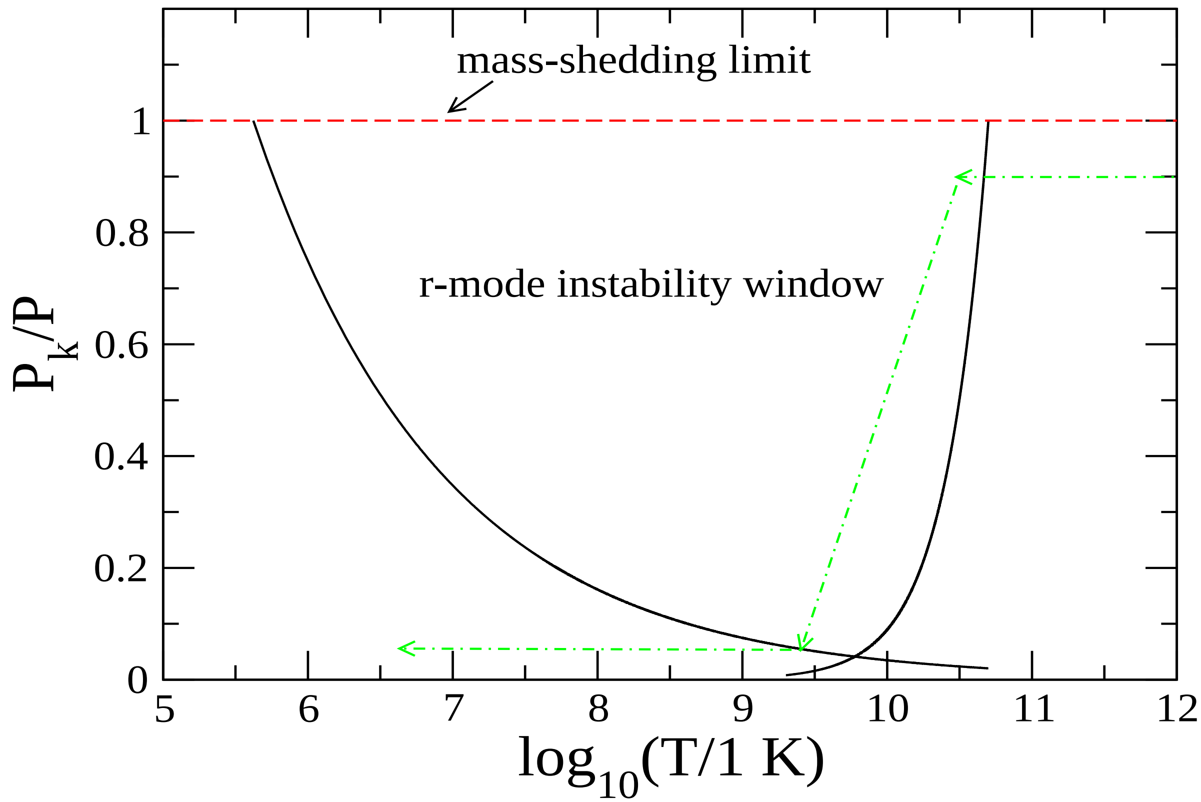
1. Limit the spin rates of neutron stars?

2. Gravitational radiation detectable?

Depend on how large α can grow up to

What if $\alpha_{\mathrm{max}} \sim O(1)$?

Evolution of a hot, young neutron star



- Above the solid curve, r-modes unstable.
- Below the solid curve, viscosity damps the instability.
- Green line: Evolution of NS (hand-drawn line!)

Can r-modes grow big enough ($\alpha \sim O(1)$) ?

- Detailed physics involved: solid crust, B-field, hyperon bulk viscosity, superfluidity,...etc. (Many uncertainties...no definitive conclusion.)
- When α is large, nonlinear hydrodynamical effects become important!

Nonlinear hydrodynamics study

- Questions:
  1. Can nonlinear effects limit the growth of r-modes?
  2. If so, what is the dominant mechanism?
  3. What is the saturation amplitude $\alpha_{\rm sat}$?
- Methods: Numerical simulations and high-order perturbative analysis
- Numerical Study:
  1. Stergioulas & Font (2001)
  2. Lindblom, Tohline, & Vallisneri (2001)
  3. Our work (2002, 2004)
- 2nd order perturbative analysis:

  Cornell's group: Arras, Flanagan, Morsink, Schenk, Teukolsky, & Wasserman (2002)

What is the challenge for 3D numerical simulations?

growth time $\gg$ rotation period

- Growth time of the r-mode ~ 40 s.
- Rotation period (dynamical timescale) of neutron stars ~ 1 ms.
- Need to evolve the system in 3D for ~ 1000 rotation periods!
- Main difficulties:
  1. Numerical solutions can accurately model $O(10-100)$ rotations only.
  2. Computational resource

     (we needed $\sim O(100\mathrm{k})$ CPU hours on NASA's supercomputers.)

Special treatments are needed!

Our numerical work

Description of the system

- We solve the 3D fully nonlinear Newtonian hydrodynamics equations.
- General relativity is treated approximately by adding a post-Newtonian gravitational-radiation reaction force to the system:

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 \\
\rho(\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v}) &= -\nabla P - \rho \nabla \Phi + \rho \vec{F}^{\mathrm{GR}}
\end{aligned}
$$

- $\vec{F}^{\mathrm{GR}}$ is responsible for the growth of the r-modes.
- Gravitational potential: $\nabla^2 \Phi = 4\pi G \rho$
- Equation of State: $P = k\rho^2$

Numerical difficulties

- Nonlinear PDE is difficult.
- Smooth initial data can produce discontinuous solutions (e.g., shock waves).
- Standard finite-difference discretization fails near discontinuities.

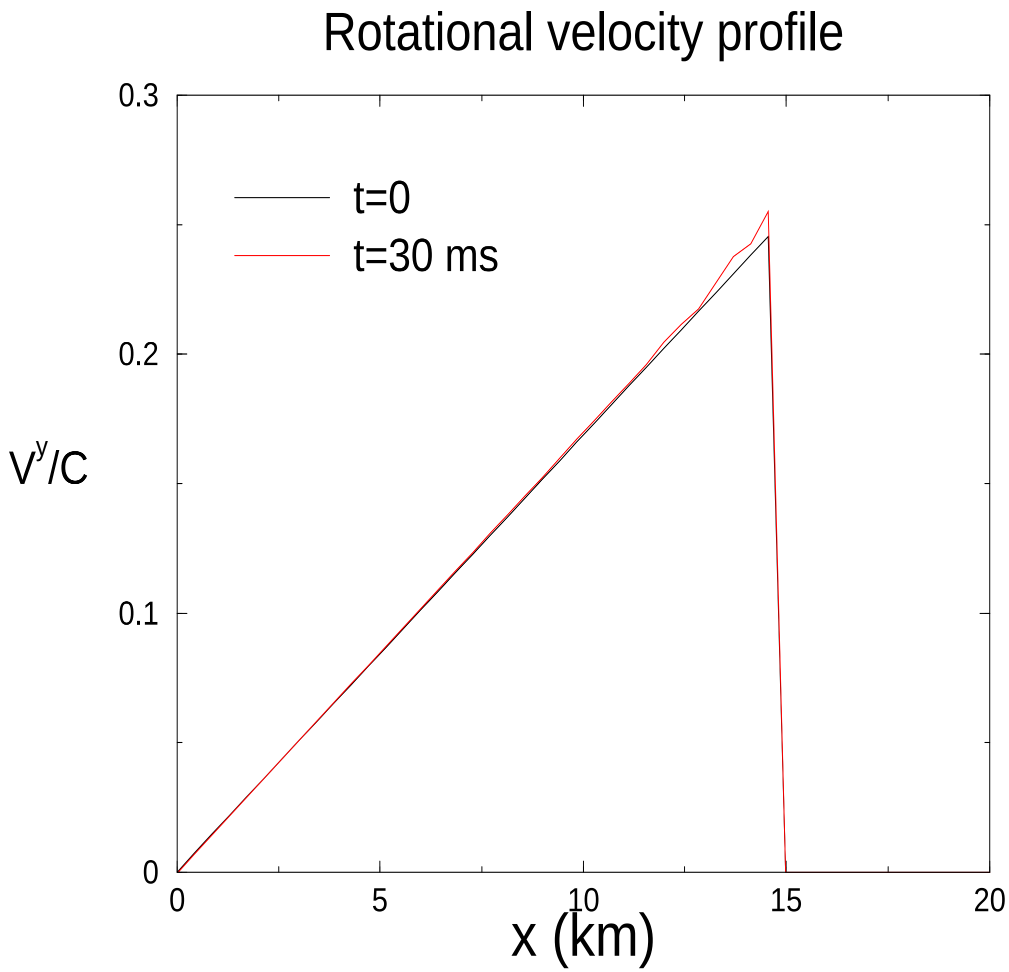
High resolution shock capturing (HRSC) scheme

- Main properties of HRSC:
  1. At least 2nd order accuracy on smooth solutions (i.e., numerical error $\sim O(\Delta x)^2$).
  2. Sharp resolution of discontinuities.
- Skip the detail....

Code validation

Code test: Equilibrium rotating star

- Star model: $M = 1.64 M_\odot$, $R_\mathrm{p} = 11$ km, $R_\mathrm{e} = 14.5$ km, $T = 1.24$ ms



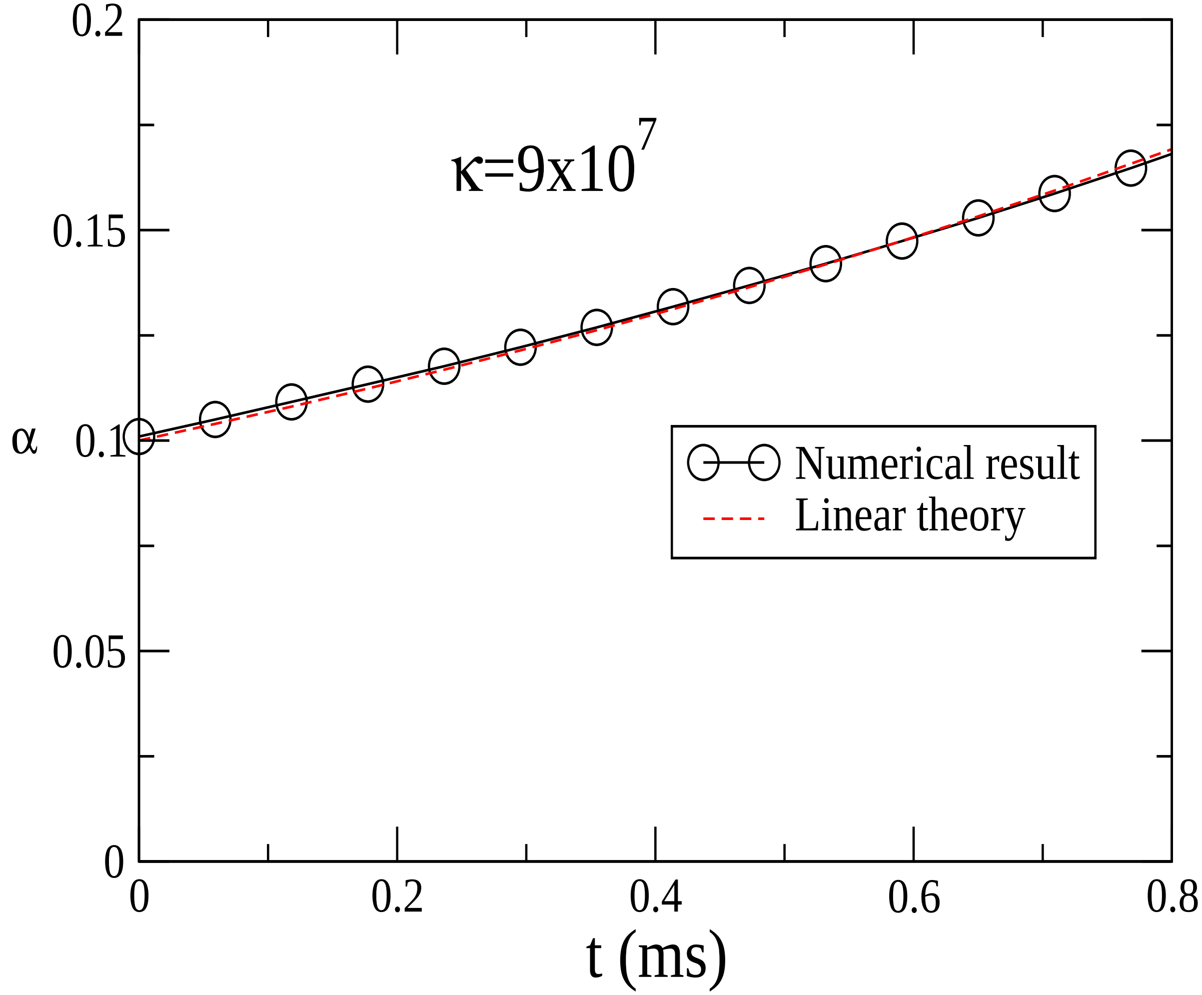
Code test: Comparison with linear theory ($\alpha \to 0, \Omega \to 0$)

- Linear theory: $\alpha = \alpha_0 e^{t/\tau}, \ \tau^{-1} = \kappa \frac{G\Omega^6}{2c^7} \left(\frac{256}{405}\right)^2 \int \rho r^4 d^3x$
- $\kappa = 1$ is the correct post-Newtonian value.
- We use a large κ to perform the test.
- We define the nonlinear r-mode amplitude by

$$\alpha(t) \equiv \frac{8\pi R |J_{22}|}{\Omega \int \rho r^4 d^3x}$$

- $J_{22} = \int \rho r^2 \vec{v} \cdot \vec{Y}_{lm}^{B*} d^3x$ is the mass-current quadrupole moment.

Code test: Evolution of α in a slowly rotating star



Physics results

- Question 1:

  How would a large-amplitude nonlinear r-mode evolve?

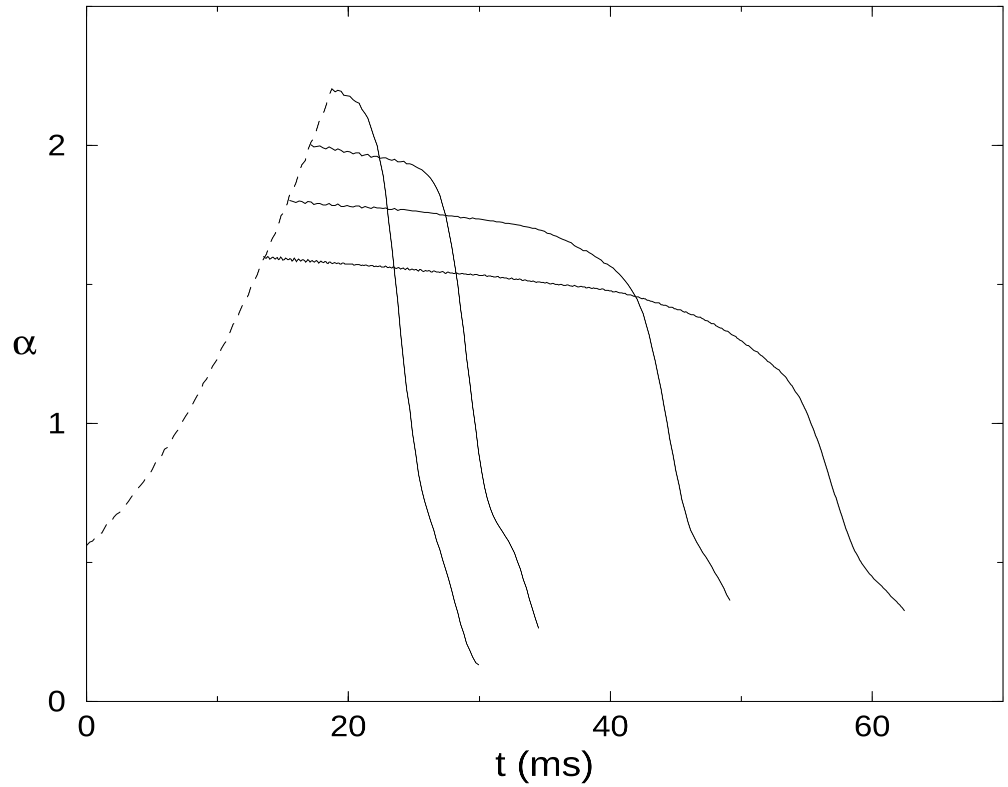
- What is a large-amplitude nonlinear r-mode?

  A configuration resulting from the growth of a small linear r-mode due to gravitational radiation reaction.

- How to create and evolve the configuration?

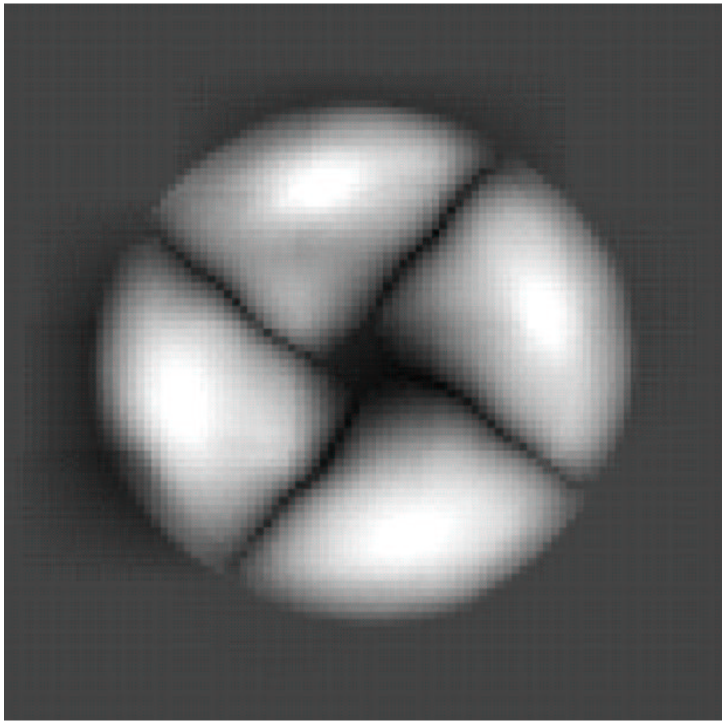
  1. To shorter the growth time, we use a post-Newtonian radiation reaction force with $\kappa = 4500$ to drive a (small) linear r-mode up to $\alpha \sim O(1)$.
  2. Growth time: 40 sec $\longrightarrow$ 10 ms.
  3. Evolve the resulting configuration with $\kappa = 1$.

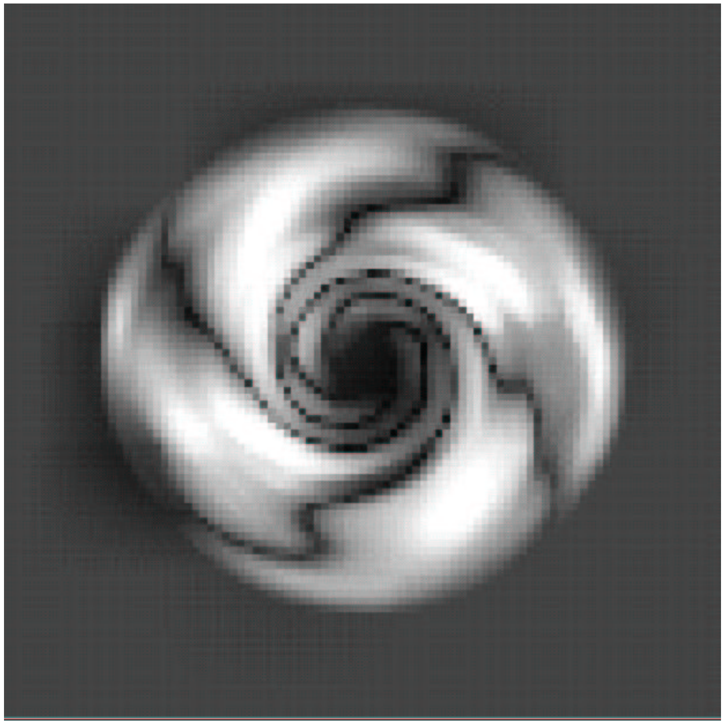
Mode amplitude α vs time (grid resolution 129^3)



α: slowly decaying $\longrightarrow$ catastrophic decay.

r-mode pattern $\bar{\alpha}$ on the equatorial plane

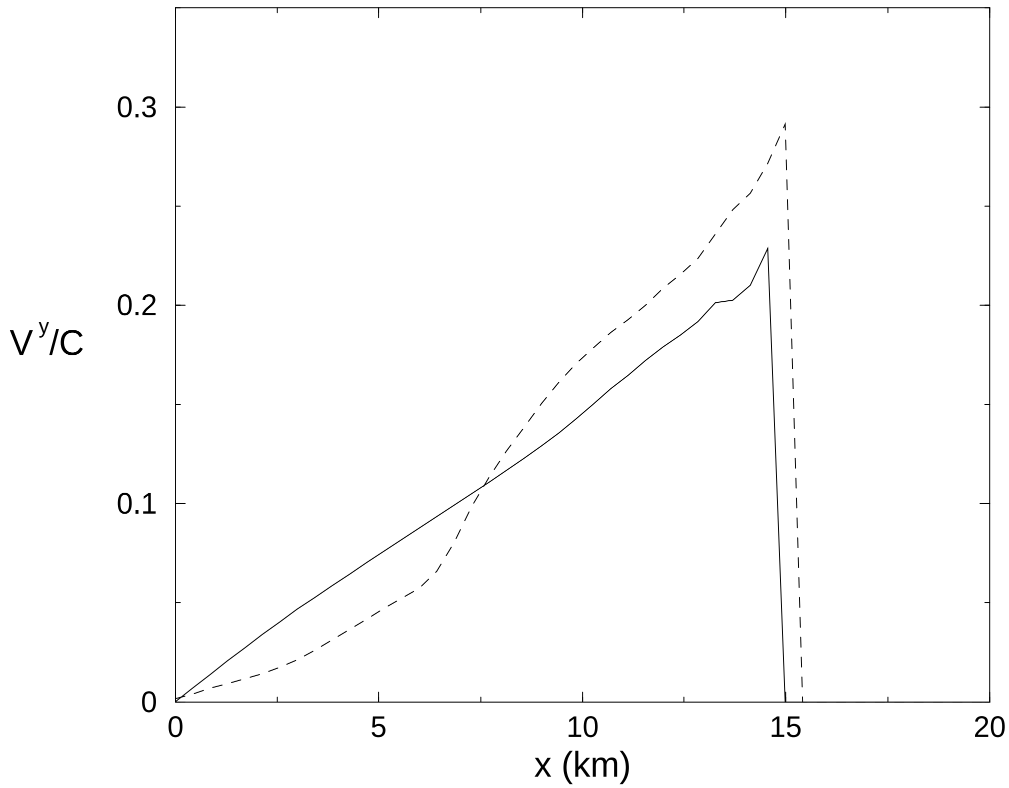




- $\alpha(t) \equiv \int \bar{\alpha}(t, \vec{x}) d^3 x$
- Before the breakdown: “regular” shape
- After the breakdown: whirlpool-like spiral

Differential rotation

Rotational velocity: v^y along the x-axis



- Solid line: Before the breakdown of α.
- Dashed line: After the breakdown of α.

- Question 2:

  What causes the catastrophic decay of α?

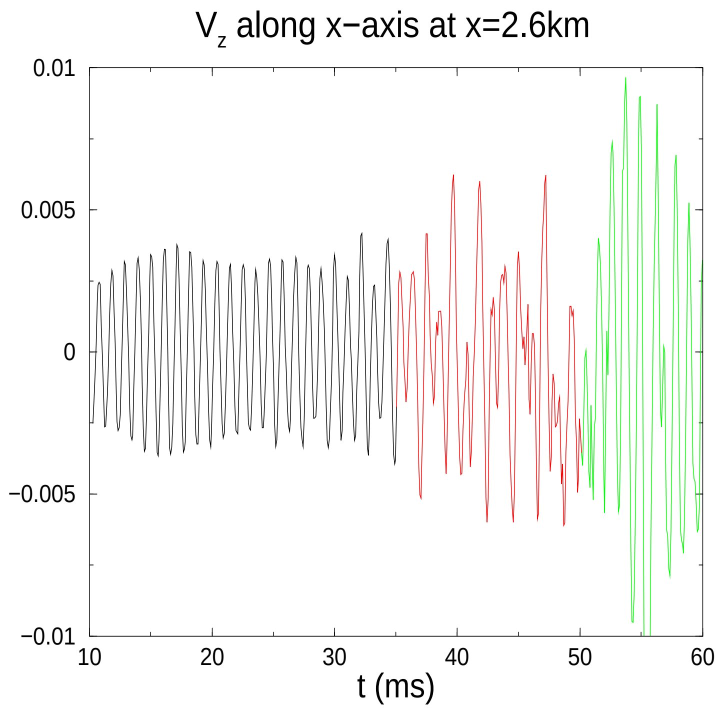
- Claim:

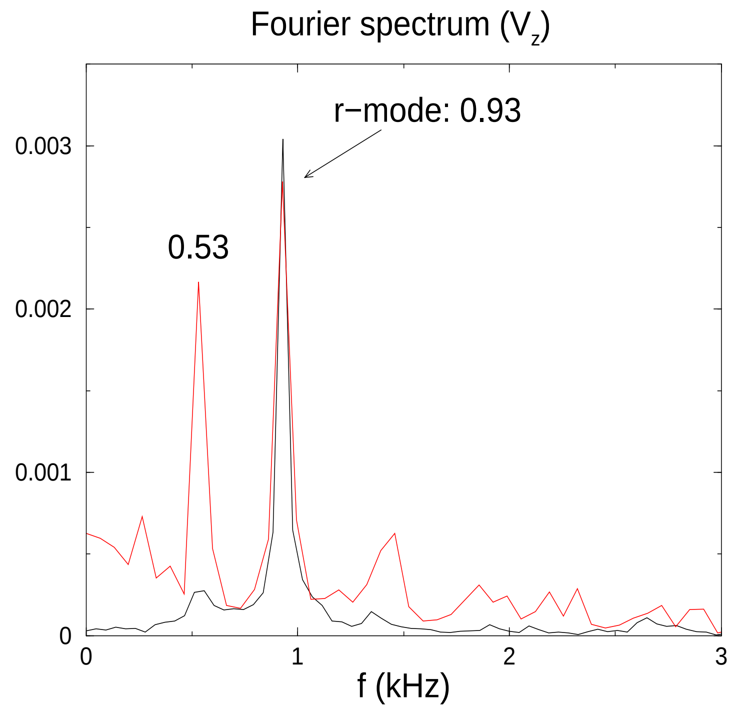
  1. The r-mode couples to other modes during the slowly decaying phase.
  2. Those other fluid modes grow to large amplitudes, leading to the collapse of α.

- How to confirm the claim?

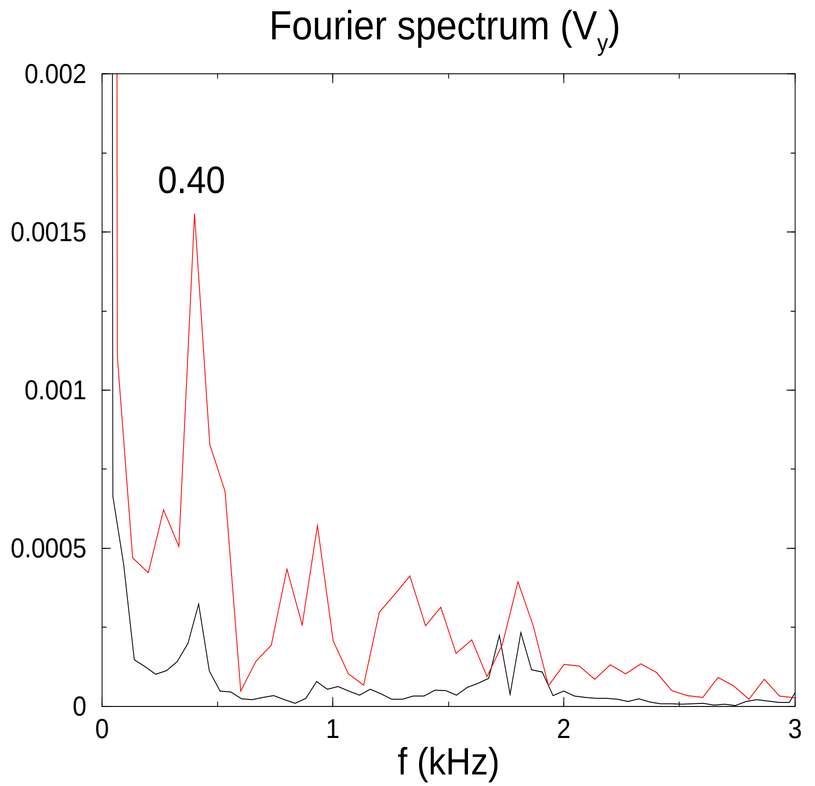
  Use Fourier spectrum analysis.

Case: $\alpha = 1.6$





- We see a significant growth of the fluid mode at 0.53 kHz.



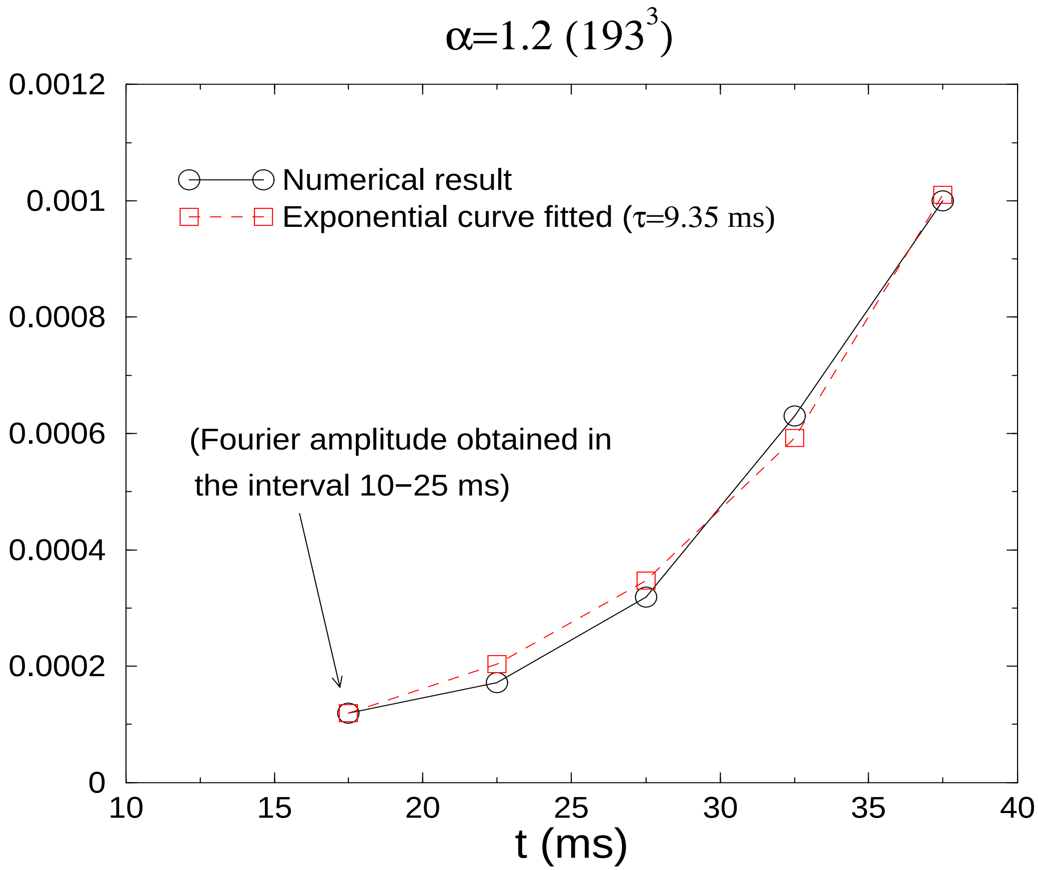
- A significant growth of the fluid mode at 0.40 kHz.

- v^z spectrum: r-mode (0.93 kHz) and mode A (0.53 kHz).
- v^y spectrum: mode B (0.40 kHz).
- The frequencies satisfy the 3-mode resonance condition:

$$\boxed{0.53 + 0.40 = 0.93}$$

- Energy transfer between 3 coupled modes is effective only if $f_A + f_B \approx f_R$.

Growth of the coupled mode at 0.53 kHz



- Their amplitudes grow exponentially.
- The coupled modes are unstable due to their couplings to the r-mode.

Summary of our numerical results

- During the initial slowly decaying phase of α, the system is dominated by a 3-mode coupling.
- When the coupled modes (A & B) grow to amplitudes $\sim \alpha$, the catastrophic decay sets in.
- The system becomes fully nonlinear:
  The uniformly rotating star changes rapidly to a differentially rotating configuration.

What about an r-mode growing from a small amplitude?

Numerical simulation + perturbative analysis

In the slowly decaying phase, the system can be described by a 3-mode system:

$$\frac{dq_R}{dt} + i\omega_R q_R = \frac{q_R}{\tau_R} + i\omega_R \kappa^*_{RAB} q^*_A q^*_B$$

$$\frac{dq_A}{dt} + i\omega_A q_A = -\gamma_A q_A + i\omega_A \kappa^*_{RAB} q^*_R q^*_B$$

$$\frac{dq_B}{dt} + i\omega_B q_B = -\gamma_B q_B + i\omega_B \kappa^*_{RAB} q^*_R q^*_A$$

- r-mode amplitude: $|q_R(t)| \approx \alpha$
- From $d\alpha/dt$ determined in our simulation, we can obtain κ_{RAB}
- We can then determine the behavior of the r-mode for small α values

- In the 3-mode system, α is governed by:

$$\frac{d\alpha}{dt} = \frac{\alpha}{\tau_R} + \frac{\omega_R}{\alpha}\mathrm{Im}(\kappa_{RAB} q_A q_B q_R)$$

- For $\alpha = 1.6$ (at high enough grid resolution):

$$\frac{d\alpha}{dt} = -2.9 \times 10^{-3}\,(\mathrm{ms})^{-1}$$

- Taking $|q_A| \approx |q_B| \equiv q \approx \alpha$:

$$|\kappa_{RAB}| \sim \frac{1}{\omega_R q^2}\left|\frac{d\alpha}{dt} - \frac{\alpha}{\tau_R}\right| \sim 3 \times 10^{-4}$$

- Ideally, $\kappa_{RAB} = \mathrm{const.}$
  We find that $|\kappa_{RAB}|$ changes only by $< 3\%$ for $\alpha = 1.2$

Saturation amplitude of the r-mode

- How an r-mode would evolve from a small initial α ?
- Assume:
  1. r-mode couples only to A and B initially.
  2. 3-mode system is valid.
  3. When A and B grow to amplitudes $q \approx \alpha$, the system becomes fully nonlinear and the r-mode is destroyed.
- This critical point is given by (setting $d\alpha/dt = 0$)

$$\alpha_{\rm crit} \sim \frac{1}{\omega_R \tau_R |\kappa_{RAB}|} \sim 0.02$$

Comparison with other's work

1. Numerical simulations (Stergioulas & Font, Lindblom et al.):
    - They suggested that large amplitude r-modes could exist for some long period of time.
    - Lindblom et al suggested that the r-mode could grow to $\alpha \approx 3.3$ before shock waves developed at the star's surface.

    Our work shows that nonlinear mode couplings can limit α to $\sim O(10^{-2})$

2. 2nd order perturbative analysis (Arras et al.):
    - r-modes could couple to many short-wavelength modes via three-mode couplings.
    - They suggested $\alpha_{\rm sat} \ll 1$.

    Our work does not confirm or rule out their analysis

Conclusions

- We show explicitly that the existence of a particular three-mode coupling as a saturation mechanism of the r-mode instability
- Once the coupled modes grow to large amplitudes ($\sim \alpha$), a catatrophic decay of the r-mode sets in.
- Our analysis suggests $\alpha_{\rm sat} \sim O(10^{-2})$, which is much smaller than that suggested by previous numerical simulations.