A new formulation for evolving neutron star spacetimes

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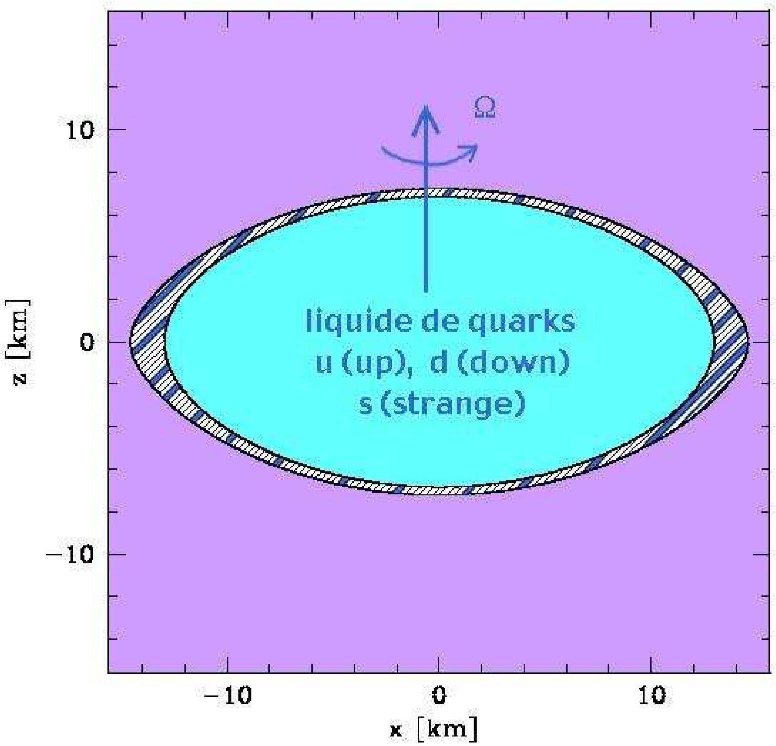
Local context (i.e. within the Meudon - Warsaw group)



[Zdunik, Haensel, Gourgoulhon, A&A **372**, 535 (2001)]

Most previous computations:
stationary models of compacts stars

- **single rotating stars:** determination of maximum mass, maximum rotation rate, ISCO frequency, accretion induced spin-up, for various models of dense matter
- **binary stars :** determination of last stable orbit (end of chirp phase in the GW signal) for neutron stars and strange quark stars

Exceptions: 1D gravitational collapse NS $\longrightarrow$ BH [in GR (1991,1993) and in tensor-scalar theories (1998)], 3D stellar core collapse [Newtonian (1993) and IWM approx. (2004)], inertial modes in rotating star [Newtonian (2002) and IWM approx. (2004)].

Computing time evolution of neutron stars

Astrophysical motivations:

- Oscillations and stability
  - ⋆ beyond the linear regime
  - ⋆ for rapidly rotating stars
- Direct computation of resulting gravitational wave emission
- Phase transitions
- Collapse of supramassive neutron stars to black hole
- Formation and stability of black hole - torus systems

Global context (i.e. studies from other groups)

Numerical studies of time evolution of rapidly rotating NS

2D (axisymmetric) codes:

- **Nakamura et al. (1981,1983) :** rotating collapse to a black hole, full GR, cylindrical coordinates (ϖ, z, φ)
- **Stark & Piran (1985) :** rotating collapse to a black hole, extraction of GW, full GR, spherical coordinates (r, θ, φ)
- **Dimmelmeier, Font & Müller (2002) :** stellar core collapse, IWM approx. to GR, spherical coordinates (r, θ, φ) [A&A **388**, 917 (2002)] [A&A **393**, 523 (2002)]
- **Shibata (2003) :** general purpose axisymmetric full GR code, Cartesian coordinates (x, y, z) + "cartoon" method [Shibata, PRD **67**, 024033 (2003)]
  $\longrightarrow$ GW from axisymmetrically oscillating NS [Shibata & Sekiguchi, PRD **68**, 104020 (2003)]
  $\longrightarrow$ GW from axisymmetric stellar core collapse to NS [Shibata & Sekiguchi, PRD **69**, 084024 (2004)]
  $\longrightarrow$ collapse of rotating supramassive NS to BH [Shibata, ApJ **595**, 992 (2003)]
  $\longrightarrow$ collapse of rapidly rotating polytopes to BH [Sekiguchi & Shibata, PRD **70**, 084005 (2004)]

Global context (i.e. studies from other groups)

Numerical studies of time evolution of rapidly rotating NS

3D codes:

- **Shibata (1999)** [Shibata, Prog. Theor. Phys. **101**, 1199 (1999)] [Shibata, PRD **60**, 104052 (1999)] : full GR, Cartesian coordinates (x, y, z)
  $\longrightarrow$ 3D collapse of rotating NS $(\gamma = 1)$ [Shibata, Baumgarte & Shapiro, PRD **61**, 044012 (2000)]
  $\longrightarrow$ binary NS merger [Shibata & Uryu, PRD **61**, 064001 (2000)], [Shibata, Taniguchi & Uryu, PRD **68**, 084020 (2003)]

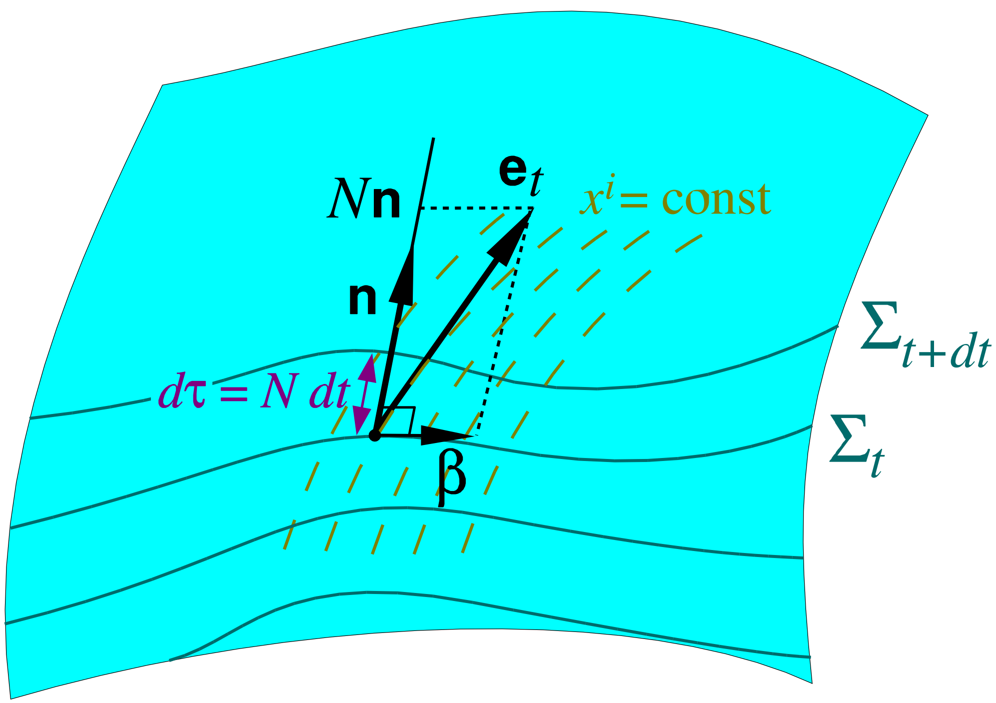
- **GR_ASTRO/Cactus code (2000,2002)** [Font et al., PRD **61**, 044011 (2000)] [Font et al., PRD **64**, 084024 (2002)] : full GR, Cartesian coordinates (x, y, z)

- **Whisky/Cactus code (2004)** [Baiotti et al., gr-qc/0403029]: full GR, Cartesian coordinates (x, y, z)

- **"Mariage des maillages" code (2004)** [Dimmelmeier, Novak, Font, Ibañez & Müller, gr-qc/0407174] : IWM approx. to GR, spherical coordinates (r, θ, φ)

Time evolution in general relativity: the 3+1 formalism

Foliation of spacetime by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$; on each hypersurface, pick a coordinate system $(x^i)_{i\in\{1,2,3\}}$
$\Longrightarrow$ $(x^\mu)_{\mu\in\{0,1,2,3\}} = (t, x^1, x^2, x^3)$ = coordinate system on spacetime



$\mathbf{n}$: future directed unit normal to Σ_t :
$\mathbf{n} = -N\,\mathbf{d}t$, N : lapse function
$\mathbf{e}_t = \partial/\partial t$: time vector of the natural basis associated with the coordinates (x^μ)

$$\left.\begin{array}{l} N \text{ : lapse function} \\ \boldsymbol{\beta} \text{ : shift vector} \end{array}\right\} \mathbf{e}_t = N\mathbf{n} + \boldsymbol{\beta}$$

Geometry of the hypersurfaces Σ_t:
– induced metric $\boldsymbol{\gamma} = \mathbf{g} + \mathbf{n} \otimes \mathbf{n}$
– extrinsic curvature : $\mathbf{K} = -\frac{1}{2}\pounds_{\mathbf{n}}\boldsymbol{\gamma}$

$$\boxed{g_{\mu\nu}\,dx^\mu\,dx^\nu = -N^2\,dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j dt)}$$

3+1 decomposition of Einstein equation

Orthogonal projection of Einstein equation onto Σ_t and along the normal to Σ_t :

- Hamiltonian constraint: $\boxed{R + K^2 - K_{ij} K^{ij} = 16\pi E}$
- Momentum constraint : $\boxed{D_j K^{ij} - D^i K = 8\pi J^i}$
- Dynamical equations :

$$\boxed{\frac{\partial K_{ij}}{\partial t} - \pounds_{\boldsymbol{\beta}} K_{ij} = -D_i D_j N + N \left[R_{ij} - 2 K_{ik} K^k_{\ j} + K K_{ij} + 4\pi ((S - E)\gamma_{ij} - 2 S_{ij}) \right]}$$

$E := \mathbf{T}(\mathbf{n}, \mathbf{n}) = T_{\mu\nu} n^\mu n^\nu$, $\quad J_i := -\gamma_i^{\ \mu} T_{\mu\nu} n^\nu$, $\quad S_{ij} := \gamma_i^{\ \mu} \gamma_j^{\ \nu} T_{\mu\nu}$, $\quad S := S_i^{\ i}$

D_i : covariant derivative associated with γ, $\quad R_{ij}$: Ricci tensor of D_i, $\quad R := R_i^{\ i}$

Kinematical relation between γ and **K**: $\boxed{\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i = 2 N K^{ij}}$

Resolution of Einstein equation $\equiv$ Cauchy problem

Free vs. constrained evolution in 3+1 numerical relativity

Einstein equations split into
$$\begin{cases} \text{dynamical equations} & \dfrac{\partial}{\partial t} K_{ij} = \ldots \\ \text{Hamiltonian constraint} & R + K^2 - K_{ij}K^{ij} = 16\pi E \\ \text{momentum constraint} & D_j K_i^{\ j} - D_i K = 8\pi J_i \end{cases}$$

- **2-D computations** (80's and 90's):
  partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
  fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams et al. (1994)

- **3-D computations** (from mid 90's): almost all based on **free evolution schemes**: BSSN, symmetric hyperbolic formulations, etc...
  $\Longrightarrow$ problem: exponential growth of constraint violating modes

Standard issue 1:	the constraints usually involve elliptic equations and 3-D elliptic solvers are CPU-time expensive !

Cartesian vs. spherical coordinates in 3+1 numerical relativity

- **1-D and 2-D computations:** massive usage of **spherical coordinates** (r, θ, φ)

- **3-D computations:** almost all based on **Cartesian coordinates** (x, y, z), although spherical coordinates are better suited to study objects with spherical topology (black holes, neutron stars). Two exceptions:
  – Nakamura et al. (1987): evolution of pure gravitational wave spacetimes in spherical coordinates (but with Cartesian components of tensor fields)
  – Stark (1989): attempt to compute 3D stellar collapse in spherical coordinates

Standard issue 2: spherical coordinates are singular at $r = 0$ and $\theta = 0$ or π !

Standard issues 1 and 2 can be overcome

Standard issues 1 and 2 are neither *mathematical* nor *physical*, but *technical* ones
$\Longrightarrow$ they can be overcome with appropriate techniques

Spectral methods allow for

- an automatic treatment of the singularities of spherical coordinates (issue 2)
- fast 3-D elliptic solvers in spherical coordinates: 3-D Poisson equation reduced to a system of 1-D algebraic equations with banded matrices [Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)] (issue 1)

Conformal metric and dynamics of the gravitational field

York (1972) : **Dynamical degrees of freedom** of the gravitational field carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3}\,\gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

$$\hat{\gamma}_{ij} = \text{tensor density of weight } -2/3$$

To work with tensor fields only, introduce an *extra structure* on Σ_t: a flat metric **f** such that $\dfrac{\partial f_{ij}}{\partial t} = 0$ and $\gamma_{ij} \sim f_{ij}$ at spatial infinity (asymptotic flatness)

Define $\boxed{\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij}}$ or $\boxed{\gamma_{ij} =: \Psi^4\,\tilde{\gamma}_{ij}}$ with $\boxed{\Psi := \left(\dfrac{\gamma}{f}\right)^{1/12}}$, $f := \det f_{ij}$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

Notations: $\tilde{\gamma}^{ij}$: inverse conformal metric : $\tilde{\gamma}_{ik}\,\tilde{\gamma}^{kj} = \delta_i^{\ j}$
$\tilde{D}_i$: covariant derivative associated with $\tilde{\gamma}_{ij}$, $\tilde{D}^i := \tilde{\gamma}^{ij}\tilde{D}_j$
$\mathcal{D}_i$: covariant derivative associated with f_{ij}, $\mathcal{D}^i := f^{ij}\mathcal{D}_j$

Dirac gauge

Conformal decomposition of the metric γ_{ij} of the spacelike hypersurfaces Σ_t:

$$\gamma_{ij} =: \Psi^4 \, \tilde{\gamma}_{ij} \qquad \text{with} \qquad \tilde{\gamma}^{ij} =: f^{ij} + h^{ij}$$

where f_{ij} is a flat metric on Σ_t, h^{ij} a symmetric tensor and Ψ a scalar field defined by $\Psi := \left(\frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12}$

Dirac gauge (Dirac, 1959) = **divergence-free** condition on $\tilde{\gamma}^{ij}$: $\boxed{\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0}$
where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.

Compare
- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Dirac gauge: discussion

- introduced by Dirac (1959) in order to fix the coordinates in some Hamiltonian formulation of general relativity; originally defined for Cartesian coordinates only: $\frac{\partial}{\partial x^j}\left(\gamma^{1/3}\gamma^{ij}\right) = 0$
  but trivially extended by us to more general type of coordinates (e.g. spherical) thanks to the introduction of the flat metric f_{ij}: $\mathcal{D}_j\left((\gamma/f)^{1/3}\gamma^{ij}\right) = 0$

- fully specifies (up to some boundary conditions) the coordinates in each hypersurface Σ_t, including the initial one $\Rightarrow$ allows for the search for stationary solutions

- leads asymptotically to transverse-traceless (TT) coordinates (same as minimal distortion gauge). Both gauges are analogous to Coulomb gauge in electrodynamics

- turns the Ricci tensor of conformal metric $\tilde{\gamma}_{ij}$ into an elliptic operator for h^{ij} $\Longrightarrow$ the dynamical Einstein equations become a **wave equation** for h^{ij}

- results in a vector elliptic equation for the shift vector β^i

3+1 Einstein equations in maximal slicing + Dirac gauge

[Bonazzola, Gourgoulhon, Grandclément & Novak, PRD in press, gr-qc/0307082 v4]

- 5 elliptic equations (4 constraints + $K=0$ condition) ($\Delta := \mathcal{D}_k \mathcal{D}^k$ = flat Laplacian):

$$\Delta N = \Psi^4 N \left[4\pi(E+S) + A_{kl}A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N \quad (N\text{=lapse function})$$

$$\begin{aligned} \Delta(\Psi^2 N) &= \Psi^6 N\left(4\pi S + \frac{3}{4}A_{kl}A^{kl}\right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) + \Psi^2\left[N\left(\frac{1}{16}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij}\right.\right. \\ &\left.\left. -\frac{1}{8}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il} + 2\tilde{D}_k\ln\Psi\,\tilde{D}^k\ln\Psi\right) + 2\tilde{D}_k\ln\Psi\,\tilde{D}^k N\right]. \end{aligned}$$

$$\begin{aligned} \Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) &= 2A^{ij}\mathcal{D}_j N + 16\pi N\Psi^4 J^i - 12NA^{ij}\mathcal{D}_j\ln\Psi - 2\Delta^i{}_{kl}NA^{kl} \\ &\quad -h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l \end{aligned}$$

3+1 equations in maximal slicing + Dirac gauge (cont'd)

- 2 scalar wave equations for two scalar potentials χ and μ :

$$-\frac{\partial^2 \chi}{\partial t^2} + \Delta \chi = S_\chi$$
$$-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu = S_\mu$$

(for expression of S_χ and S_μ see [Bonazzola, Gourgoulhon, Grandclément & Novak, PRD in press, gr-qc/0307082 v4])

The remaining 3 degrees of freedom are fixed by the Dirac gauge:

(i) From the two potentials χ and μ, construct a TT tensor $\bar{h}^{ij}$ according to the formulas (components with respect to a spherical **f**-orthonormal frame)

$$\bar{h}^{rr} = \frac{\chi}{r^2}, \quad \bar{h}^{r\theta} = \frac{1}{r}\left(\frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi}\right), \quad \bar{h}^{r\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta}\right), \text{ etc...}$$

with $\Delta_{\theta\varphi}\eta = -\partial\chi/\partial r - \chi/r$

Recovering the conformal metric $\tilde{\gamma}_{ij}$ from the TT tensor $\bar{h}^{ij}$

(ii) h^{ij} is uniquely determined by the TT tensor $\bar{h}^{ij}$ as the following divergence-free (Dirac gauge) tensor :

$$h^{ij} = \bar{h}^{ij} + \frac{1}{2}\left(h\, f^{ij} - \mathcal{D}^i \mathcal{D}^j \phi\right) \qquad (1)$$

where $h := f_{ij}h^{ij}$ is the trace of h^{ij} with respect to the flat metric and ϕ is the solution of the Poisson equation $\Delta\phi = h$. The trace h is determined in order to enforce the condition $\det \tilde{\gamma}_{ij} = \det f_{ij}$ (definition of Ψ) by

$$\begin{aligned} h \;=\; & -h^{rr}h^{\theta\theta} - h^{rr}h^{\varphi\varphi} - h^{\theta\theta}h^{\varphi\varphi} + (h^{r\theta})^2 + (h^{r\varphi})^2 + (h^{\theta\varphi})^2 - h^{rr}h^{\theta\theta}h^{\varphi\varphi} \\ & -2h^{r\theta}h^{r\varphi}h^{\theta\varphi} + h^{rr}(h^{\theta\varphi})^2 + h^{\theta\theta}(h^{r\varphi})^2 + h^{\varphi\varphi}(h^{r\theta})^2 \qquad (2) \end{aligned}$$

Equations (1) and (2) constitute a coupled system which can be solved by iterations (starting from $h^{ij} = \bar{h}^{ij}$), at the price of solving the Poisson equation $\Delta\phi = h$ at each step. In practise a few iterations are sufficient to reach machine accuracy.

(iii) Finally $\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$.

Numerical implementation

Numerical code based on the C++ library LORENE (http://www.lorene.obspm.fr) with the following main features:

- **multidomain spectral methods** based on spherical coordinates (r, θ, φ), with compactified external domain ($\Longrightarrow$ spatial infinity included in the computational domain for elliptic equations)

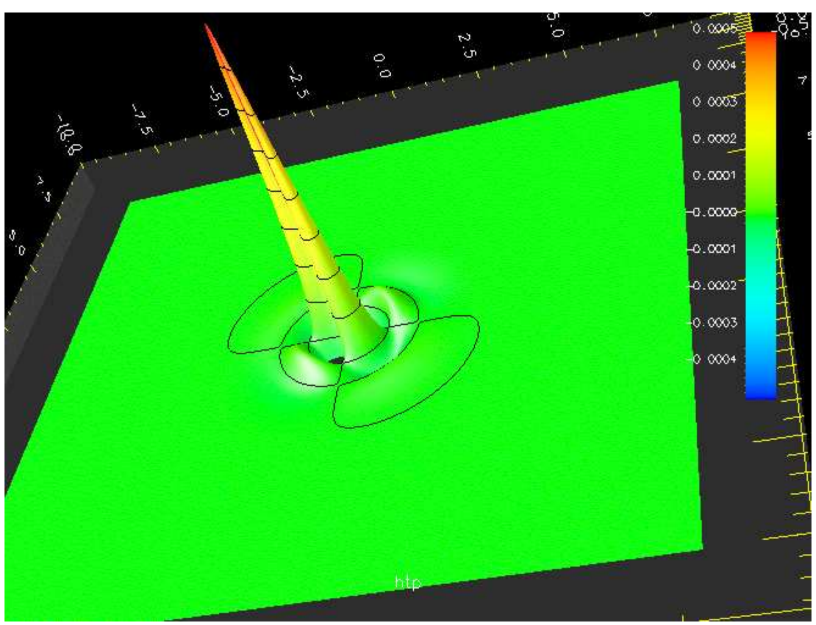
- very efficient **outgoing-wave boundary conditions**, ensuring that all modes with spherical harmonics indices $\ell = 0$, $\ell = 1$ and $\ell = 2$ are perfectly outgoing
  [Novak & Bonazzola, J. Comp. Phys. **197**, 186 (2004)]
  (*recall:* Sommerfeld boundary condition works only for $\ell = 0$, which is too low for gravitational waves)

Results on a pure gravitational wave spacetime

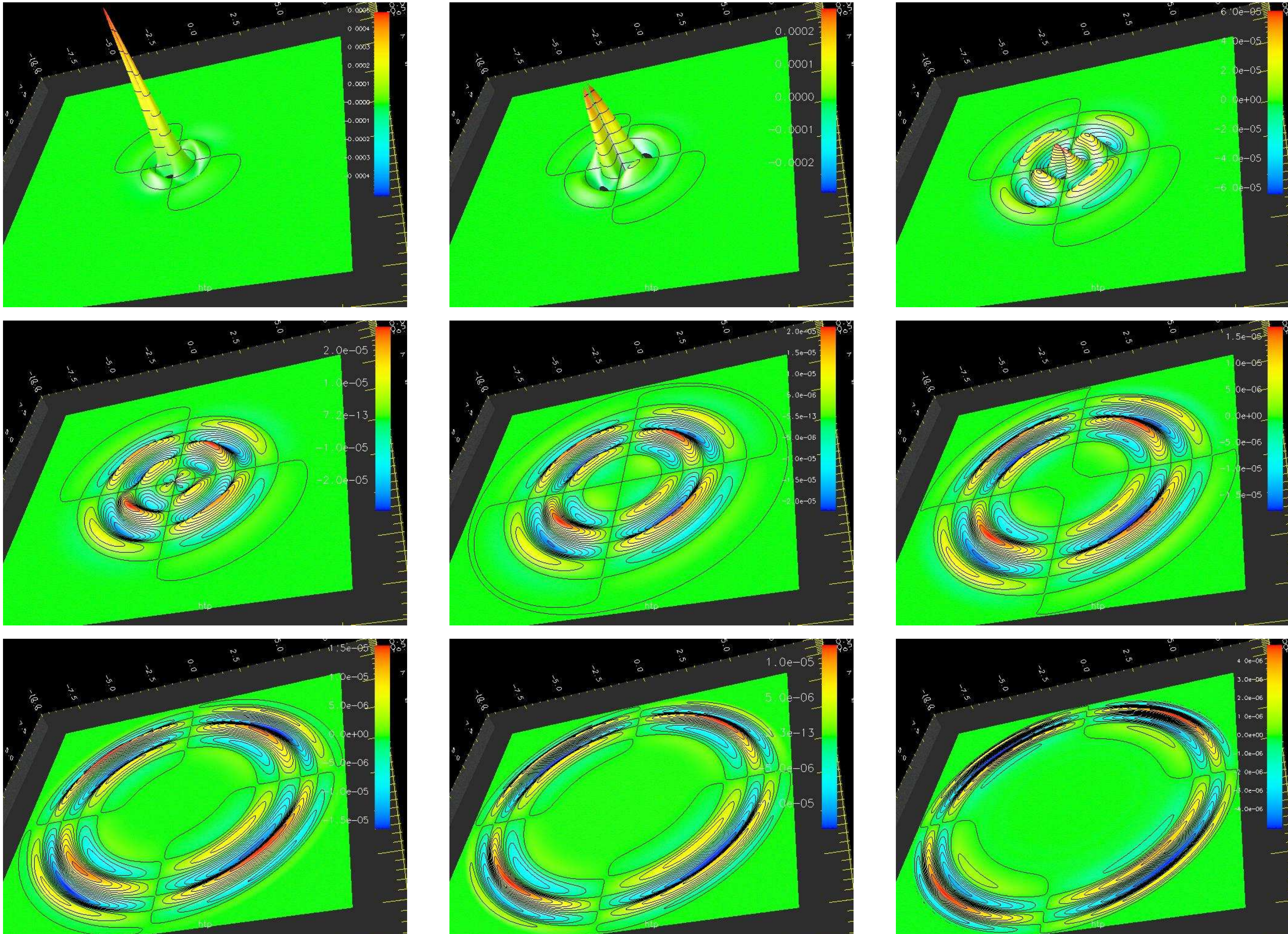
Initial data: similar to [Baumgarte & Shapiro, PRD **59**, 024007 (1998)], namely a momentarily static ($\partial\tilde{\gamma}^{ij}/\partial t = 0$) Teukolsky wave $\ell = 2$, $m = 2$:

$$\begin{cases} \chi(t=0) = \frac{\chi_0}{2} r^2 \exp\left(-\frac{r^2}{r_0^2}\right) \sin^2\theta \, \sin 2\varphi \\ \mu(t=0) = 0 \end{cases} \quad \text{with} \quad \chi_0 = 10^{-3}$$

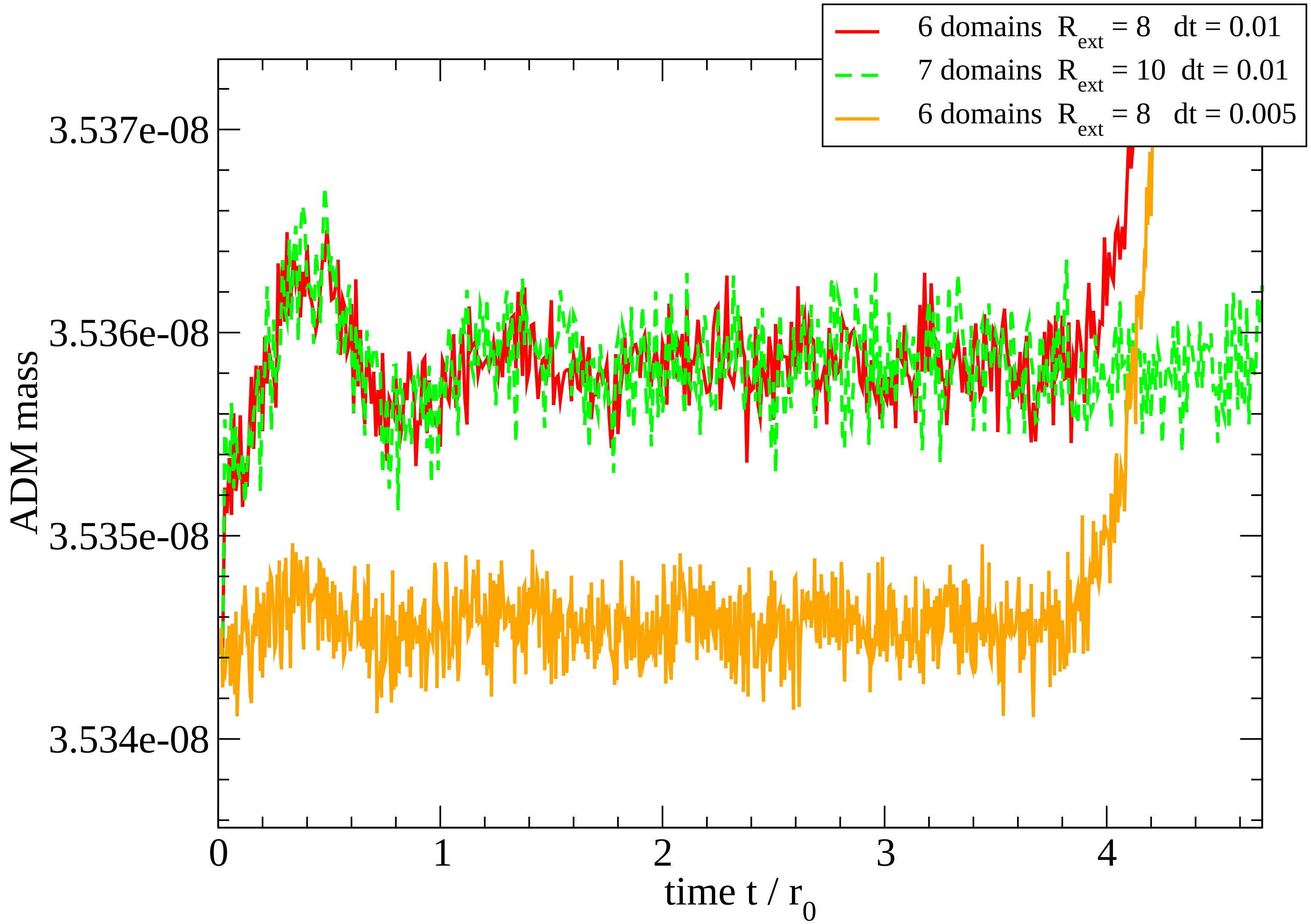
Preparation of the initial data by means of the conformal thin sandwich procedure



Evolution of $h^{\varphi\varphi}$ in the plane $\theta = \frac{\pi}{2}$



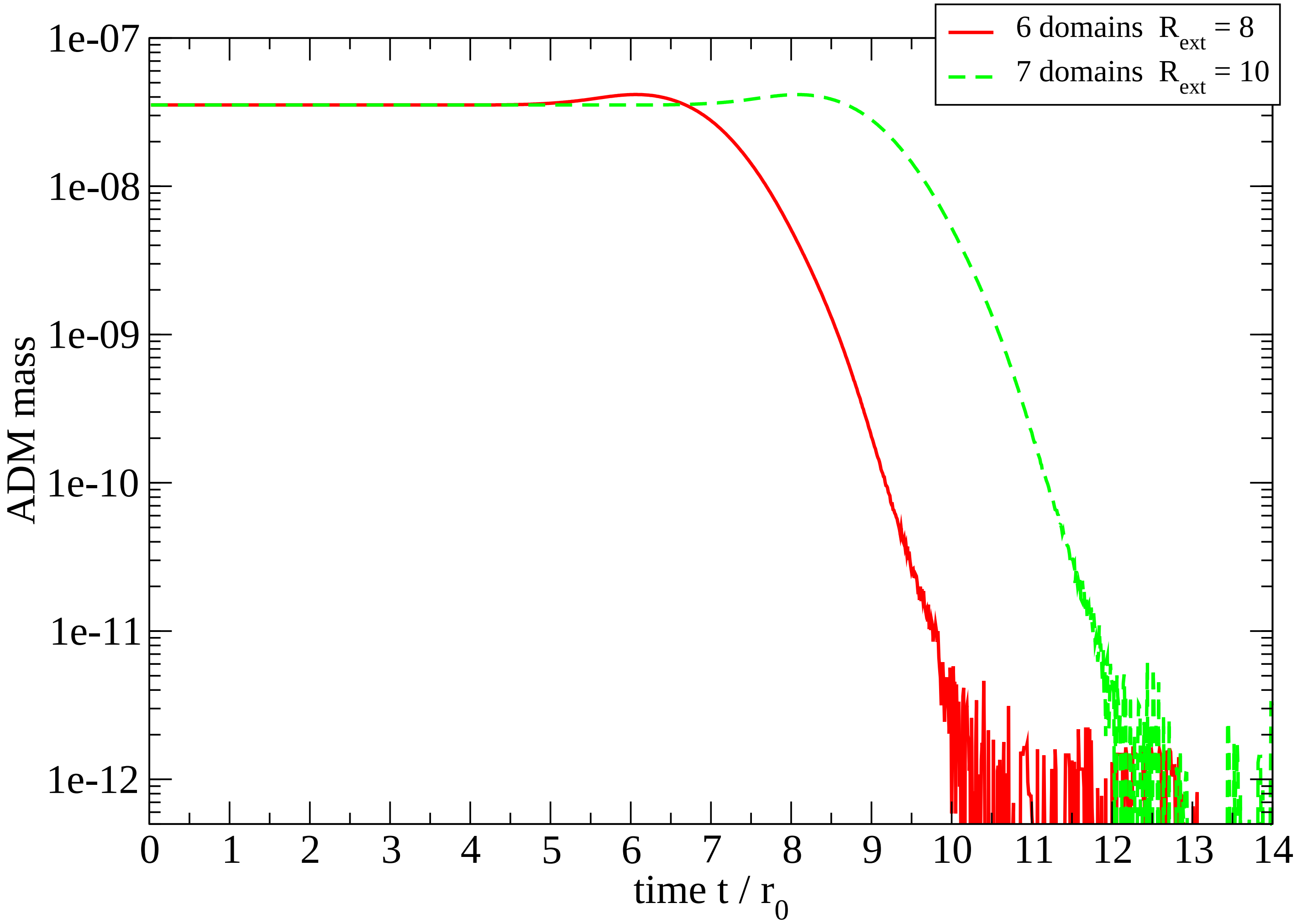
Test: conservation of the ADM mass



Number of coefficients in each domain: $N_r = 17$, $N_\theta = 9$, $N_\varphi = 8$

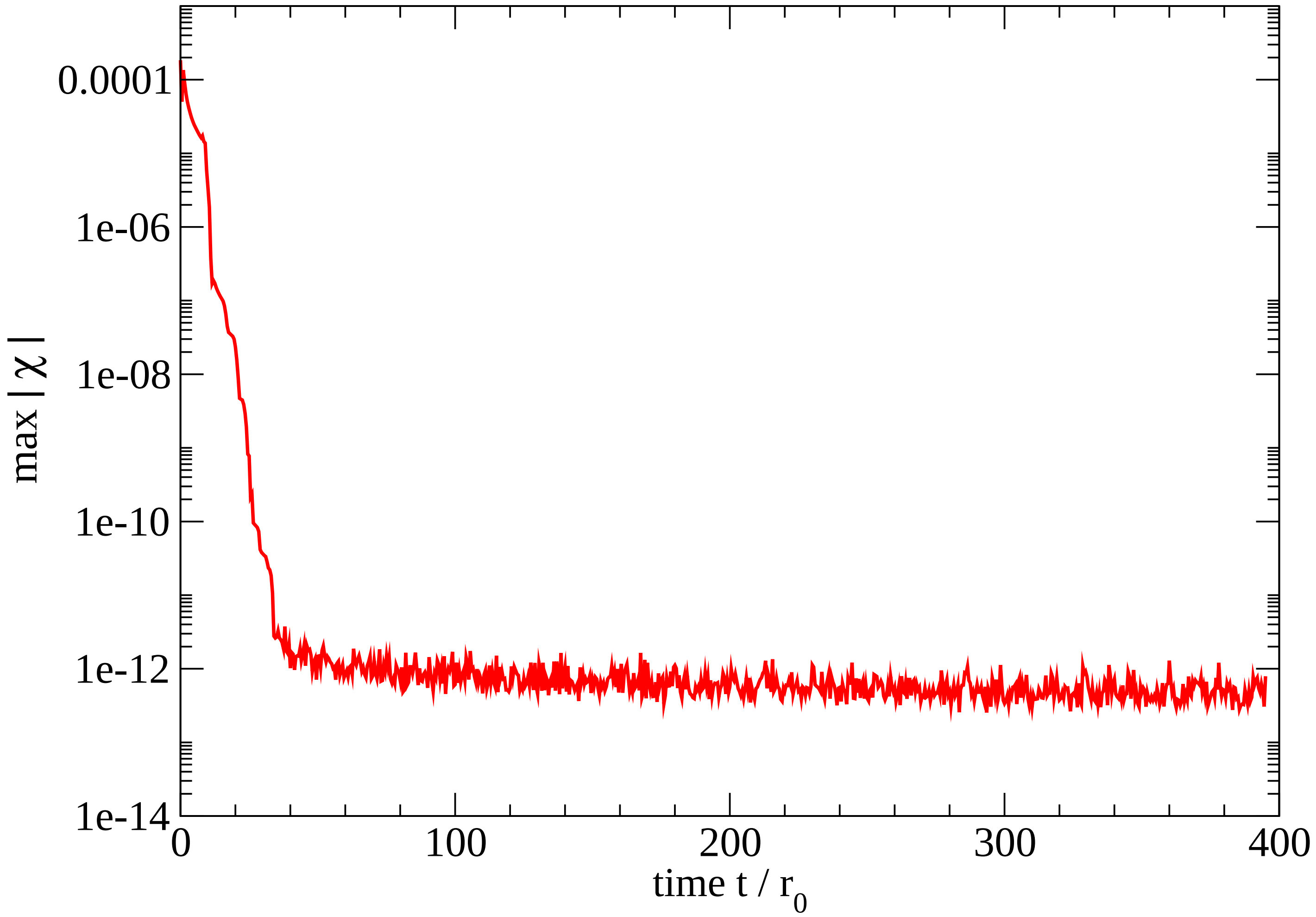
For $dt = 5\,10^{-3} r_0$, the ADM mass is conserved within a relative error lower than 10^{-4}

Late time evolution of the ADM mass



At $t > 10\, r_0$, the wave has completely left the computation domain
$\Longrightarrow$ Minkowski spacetime

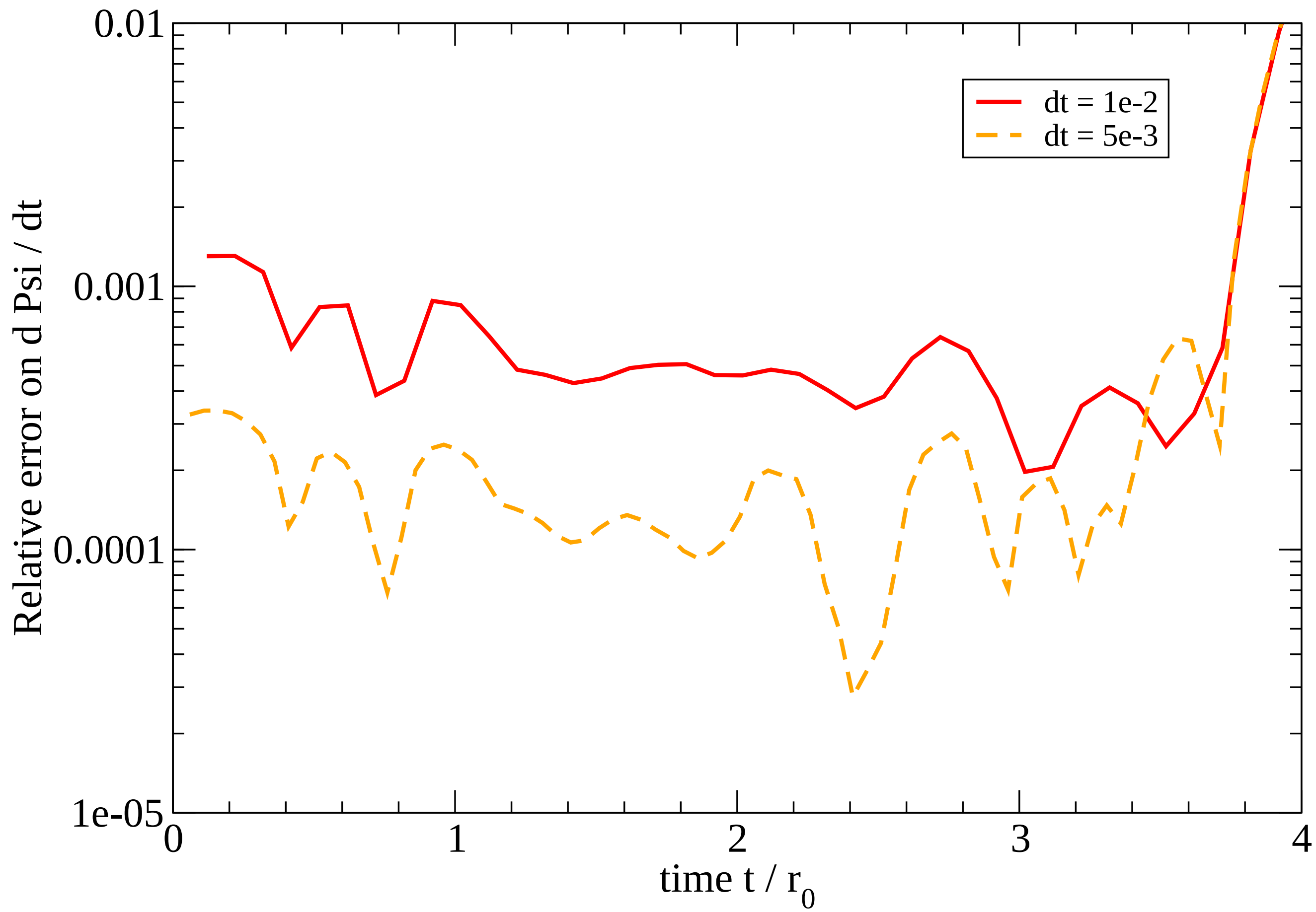
Long term stability



Nothing happens until the run is switched off at $t = 400\, r_0$!

Another test: check of the $\frac{\partial \Psi}{\partial t}$ relation

The relation $\frac{\partial}{\partial t} \ln \Psi - \beta^k \mathcal{D}_k \ln \Psi = \frac{1}{6} \mathcal{D}_k \beta^k$ (trace of the definition of the extrinsic curvature as the time derivative of the spatial metric) is not enforced in our scheme.
$\Longrightarrow$ This provides an additional test:



Summary

- Dirac gauge + maximal slicing reduces the Einstein equations into a system of
  – two scalar elliptic equations (including the Hamiltonian constraint)
  – one vector elliptic equations (the momentum constraint)
  – two scalar wave equations (evolving the two dynamical degrees of freedom of the gravitational field)

- The usage of spherical coordinates and spherical components of tensor fields is crucial in reducing the dynamical Einstein equations to two scalar wave equations

- The unimodular character of the conformal metric ($\det \tilde{\gamma}_{ij} = \det f_{ij}$) is ensured in our scheme

- First numerical results show that Dirac gauge + maximal slicing seems a promising choice for stable evolutions of 3+1 Einstein equations and gravitational wave extraction

- It remains to be tested on black hole spacetimes !

Advantages for NS spacetimes

- Spherical coordinates (inherent to the new formulation) are well adapted to the description of stellar objects (axisymmetry limit is immediate)

- Far from the central star, the time evolved quantities (h^{ij}) are nothing but the gravitational wave components in the TT gauge $\Longrightarrow$ easy extraction of gravitational radiation

- Isenberg-Wilson-Mathews approximation (widely used for equilibrium configurations of binary NS) is easily recovered in our scheme, by setting $h^{ij} = 0$

- Dirac gauge fully fixes the spatial coordinates $\Longrightarrow$ along with the resolution of constraints within the scheme, this allows for getting stationary solutions within the very same scheme, simply setting $\partial/\partial t = 0$ in the equations

A drawback: the quasi-isotropic coordinates usually used to compute stationary configurations of rotating NS do not belong to Dirac gauge, except for spherical symmetry

Future prospects

- Evolution of the gravitational field part (Einstein equations) is already implemented in LORENE (classes `Evolution` and `Tsclice_dirac_max`)
- Implementation of the hydrodynamic equations (L. Villain)
- A first step: computation of stationary configurations of rotating stars within Dirac gauge (L.-M. Lin)
- Dynamical evolution of unstable rotating stars
- Gravitational collapse
- etc...