Chapter 8

Symmetries of Riemann spaces, invariance of tensors.

8.1 Symmetry transformations.

Let $F : M_n \to M_n$ be a transformation of M_n into itself, $p \in M_n$ and $F(p) = p' \in M_n$. Then, vectors and tensors defined at the point p are carried to the point p' (see Fig. 8.1). Thereby, a tensor T that was attached to p before the transformation becomes T' attached to p'. Now consider a subset $U \subset M_n$ and its image $F(U) \subset M_n$. Suppose that F is an element of a continuous group of transformations $\{F_t\}$, with $\{F_{t_0}\}$ being the identity transformation. If $F = F_{t_1}$ and $|t_1 - t_0|$ is sufficiently small, then $F(U) \cap U \neq \emptyset$. So let $p, p' \in F(U) \cap U$. Then p is an image of another point q, $p = F(q)$, and the tensors that were attached to q before the transformation were sent into p (see Fig. 8.1). Hence, we have two tensors attached to each point p: $T(p)$ that was there before the transformation and $T'(p)$ that was sent to p from q by the transformation. The latter can be calculated from $T(q)$ by (3.10). Consequently, we can compare $T'(p)$ with $T(p)$.

This reasoning applies to any transformations, in particular to coordinate transformations if they are interpreted as "active" (i.e. as mappings of M_n into itself).

If it so happens that $T'(p) = T(p)$ for all points of the manifold, then we call the tensor field T **invariant under the action of** F, and we call F an **invariance transformation of** T. If M_n is a Riemann space, and the metric tensor of M_n is invariant under F, then the mapping F is called a **symmetry** or an **isometry** of M_n.

8.2 The Killing equations.

Let Γ be a 1-parameter family of transformations of a manifold M_n into itself such that to every value of the parameter t from the range $[t_1, t_2] \overset{\text{def}}{=} B \subset \mathbb{R}^1$ there corresponds a transformation $f_t : M_n \to M_n$:

$$\mathbb{R}^1 \supset B \overset{\text{def}}{=} [t_1, t_2] \ni t \to x'^{\alpha} = f^{\alpha}(t, \{x\}), \tag{8.1}$$

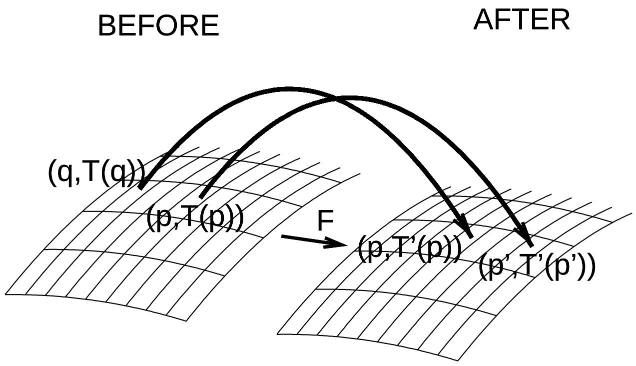

Figure 8.1: A transformation F of the manifold M_n into itself takes the point q to p, and the tensors $T(p)$ and $T(q)$ are transformed into $T'(p')$ and $T'(p)$, respectively. Thus, after the transformation we have two tensors at the same point: $T(p)$ that was there before, and $T'(p)$ that was brought into p by the transformation. If $T'(p) = T(p)$ for all $p \in M_n$, then the tensor field T is invariant under the transformation F.

where f_t is the collection of all the functions f^α at a given t. Let us also assume that for $t = t_0$ where $t_1 \le t_0 \le t_2$ the transformation f_{t_0} is an identity, i. e.

$$f^\alpha(t_0, \{x\}) = x^\alpha. \tag{8.2}$$

Example: Let $B = [0, 2\pi]$, $M_n = \mathbb{R}^3$, and let f_t be the rotation of $\mathbb{R}^3$ around a fixed axis A by the angle t. Γ is then the collection of rotations of $\mathbb{R}^3$ around A by all angles in the range $0 \le t < 2\pi$ and $t_0 = 0$.

Now apply to a $p \in M_n$ the transformations f_t corresponding to all $t \in B$. The collection of all images of p will then be an arc of a curve in M_n passing through $p = f_{t_0}(p)$, and each $p \in M_n$ may be used to generate such an arc. The arc is called the **orbit of p under the action of** Γ, and p is called the initial point of the orbit (although in "practical" instances the orbits are closed or infinite curves with no endpoints).

We assume that (1) the functions $f^\alpha(t, \{x\})$ have continuous second derivatives by t (are of class C^2); (2) each f_t is invertible; and (3) its inverse (denoted f_t^{-1}) is also of class C^2. (1) implies that along each orbit a field of tangent vectors exists and is continuously differentiable. (2) implies that the transformations of the family Γ and their inverses form a group G. The group multiplication is the superposition of the transformations:

$$\left(f_{t_2} \circ f_{t_1}\right)(\{x\}) = f_{t_2}\left(f_{t_1}(\{x\})\right), \tag{8.3}$$

where f_{t_1} is represented by $f^\alpha(t_1, \{x\})$ and f_{t_2} is represented by $f^\alpha\left(t_2, f_{t_1}(\{x\})\right)$. Assumption (3) guarantees that the orbits generated by G will have a continuously differentiable field of tangent vectors. For each transformation f_t we may then write (from Taylor's formula):

$$x'^\alpha = x^\alpha + \left.\frac{\partial f^\alpha}{\partial t}\right|_{t=t_0} (t - t_0) + O\left(\epsilon^2\right), \tag{8.4}$$

where $x'^{\alpha} = f^{\alpha}(t, \{x\})$, $x^{\alpha} = f^{\alpha}(t_0, \{x\})$, $\epsilon \stackrel{\text{def}}{=} t - t_0$ and $O(\epsilon^2)$ has the property

$$\lim_{\epsilon \to 0} \frac{O(\epsilon^2)}{\epsilon} = 0. \tag{8.5}$$

We are not making any approximation here; eq. (8.4) is exact, but the form of $O(\epsilon^2)$ will in the end turn out to be irrelevant. All such irrelevant terms will be denoted by the same symbol $O(\epsilon^2)$ even though they may not be identical to each other. The quantities

$$k^{\alpha} \stackrel{\text{def}}{=} \left.\frac{\partial f^{\alpha}}{\partial t}\right|_{t=t_0} \tag{8.6}$$

are components of the vector field tangent to the orbits at their initial points and are called the generators of the group G.

Suppose now that a tensor field $T_{\alpha\beta}$ is invariant under all the transformations in Γ:

$$T'_{\alpha\beta}(p) = T_{\alpha\beta}(p) \qquad \text{for all } p \in M_n \text{ and } t \in B. \tag{8.7}$$

What analytic condition must $T_{\alpha\beta}$ fulfil? Again from Taylor's formula:

$$T'_{\alpha\beta}(p') = T'_{\alpha\beta}(p) + \epsilon T'_{\alpha\beta,\mu}(p) k^{\mu} + O(\epsilon^2), \tag{8.8}$$

where p' has the coordinates $x'^{\alpha} = f^{\alpha}(t, \{x\})$ and p has the coordinates $x^{\alpha} = f^{\alpha}(t_0, \{x\})$. From (8.4) and (8.6) we have

$$\begin{aligned} \frac{\partial x^{\mu}}{\partial x'^{\alpha}} &= \frac{\partial}{\partial x'^{\alpha}} \left[x'^{\mu} - \epsilon k^{\mu} - O\left(\epsilon^2\right) \right] = \delta^{\mu}{}_{\alpha} - \epsilon k^{\mu}{}_{,\rho} \frac{\partial x^{\rho}}{\partial x'^{\alpha}} - O\left(\epsilon^2\right) \\ &= \delta^{\mu}{}_{\alpha} - \epsilon k^{\mu}{}_{,\rho} \left(\delta^{\rho}{}_{\alpha} - \epsilon k^{\rho}{}_{,\sigma} \frac{\partial x^{\sigma}}{\partial x'^{\alpha}} - O(\epsilon^2) \right) - O(\epsilon^2) \\ &= \delta^{\mu}{}_{\alpha} - \epsilon k^{\mu}{}_{,\alpha} + O(\epsilon^2). \end{aligned} \tag{8.9}$$

(ϵ is not differentiated because we advanced along the orbits by the fixed parameter distance $\epsilon = t - t_0$ which is coordinate-independent). Hence, from (8.9) and (3.10),

$$\begin{aligned} T'_{\alpha\beta}(p') &= \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} T_{\mu\nu}(p) \\ &= T_{\alpha\beta}(p) - \epsilon k^{\mu}{}_{,\alpha} T_{\mu\beta}(p) - \epsilon k^{\nu}{}_{,\beta} T_{\alpha\nu}(p) + O(\epsilon^2). \end{aligned} \tag{8.10}$$

Comparing (8.8) with (8.10) and using the invariance condition (8.7) we have

$$\epsilon \left(k^{\mu} T_{\alpha\beta,\mu} + k^{\mu}{}_{,\alpha} T_{\mu\beta} + k^{\nu}{}_{,\beta} T_{\alpha\nu} \right) + O(\epsilon^2) = 0. \tag{8.11}$$

We now divide (8.11) by ϵ, let $\epsilon \to 0$ and recall the definition of $O(\epsilon^2)$. The result is

$$k^{\mu} T_{\alpha\beta,\mu} + k^{\mu}{}_{,\alpha} T_{\mu\beta} + k^{\mu}{}_{,\beta} T_{\alpha\mu} = 0. \tag{8.12}$$

These are the **Killing equations**, and their solutions $k^{\mu}(\{x\})$ are **Killing vector fields**. Every field of tangent vectors to orbits of invariance transformations of $T_{\alpha\beta}$ must fulfil

(8.12), and every solution of (8.12) generates an invariance group of $T_{\alpha\beta}$. How to find the invariance transformations given k^α and vice versa will be shown in the next section.

Equation (8.12) can be rewritten in an equivalent, explicitly covariant form:

$$k^\mu T_{\alpha\beta;\mu} + k^\mu{}_{;\alpha} T_{\mu\beta} + k^\mu{}_{;\beta} T_{\alpha\mu} = 0. \tag{8.13}$$

If $T_{\alpha\beta} = g_{\alpha\beta}$ (the metric tensor), then, in view of $g_{\alpha\beta;\gamma} = 0$, eq. (8.13) may be rewritten as

$$k_{\alpha;\beta} + k_{\beta;\alpha} \equiv 2k_{(\alpha;\beta)} = 0. \tag{8.14}$$

In this form, the Killing equations are most easy to remember, but less convenient to work with, and apply only to the metric tensor.

Equation (8.12) applies only to a doubly covariant tensor field, and only in this case are the generators of invariances called Killing vector fields. We shall deal with invariances of other tensor fields in Sec. 8.5 and thereafter.

8.3 The connection between generators and the invariance transformations.

If ${x'}^\alpha = f^\alpha(t, \{x\})$ is a family of invariance transformations of a certain tensor field, then the corresponding generator is given by (8.6), where $t = t_0$ defines the identity transformation.

Finding the family of invariances given k^α is less straightforward. Any orbit $B \ni t \to y^\alpha(t) = f^\alpha(t, \{x\})$ is tangent to $k^\alpha(p')$ at its every point (where $\{y\}$ are the coordinates of p'). Hence the orbits must obey

$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = k^\alpha(\{y\}) \tag{8.15}$$

with the initial conditions

$$y^\alpha\big|_{t=t_0} = x^\alpha. \tag{8.16}$$

Eqs. (8.15) – (8.16) are to be understood as follows. A solution to (8.15) will be a family of curves $y^\alpha = f^\alpha(t, C_1, \ldots, C_n)$, labelled by n parameters $(C_1, \ldots, C_n)$. The condition (8.16) allows one to express the constants C_α in terms of the coordinates x^α of the initial points of the integral curves of the field k^α. In this way, we obtain the set of functions

$$y^\alpha = f^\alpha(t, \{x\}) \tag{8.17}$$

that satisfies Eqs. (8.15) and the initial condition (8.16).

8.4 Finding the Killing vector fields.

The Killing equations are applied to two kinds of problems:

1. Finding the metric tensor of a Riemann space whose symmetries are assumed – then they determine $g_{\alpha\beta}$, with k^α given.

2. Finding the symmetries of a Riemann space whose metric tensor is given – then they determine k^α with $g_{\alpha\beta}$ given.

An example of the first application will be shown in Section 8.8. The second application requires additional explanation. The Killing equations are linear and homogeneous in k^α, which means that if k^α and l^α are Killing fields, then so is $(Ak^\alpha + Bl^\alpha)$, where A and B are arbitrary constants. A general solution of the Killing equations should thus be a linear combination of basis solutions.

Does there exist a finite basis in the space of solutions of the Killing equations? The answer is: yes, but only for the proper Killing equations, i.e. for the generators of invariances of the metric tensor. The proof given below (borrowed from [9]) does not work if K_α generates an invariance group of a tensor field other than the metric tensor (and examples of infinite bases are known [10]).

For a field K_α generating symmetries of M_n we have from (8.14)

$$K_{\alpha;\beta} = -K_{\beta;\alpha}, \tag{8.18}$$

and from the Ricci identity

$$K_{\alpha;\beta\gamma} - K_{\alpha;\gamma\beta} = R^\rho{}_{\alpha\beta\gamma} K_\rho. \tag{8.19}$$

Because $R^\rho{}_{[\alpha\beta\gamma]} \equiv 0$, we have from the above

$$\left(K_{\alpha;\beta} - K_{\beta;\alpha}\right)_{;\gamma} + \left(K_{\gamma;\alpha} - K_{\alpha;\gamma}\right)_{;\beta} + \left(K_{\beta;\gamma} - K_{\gamma;\beta}\right)_{;\alpha} = 0. \tag{8.20}$$

Using now (8.18), the above reduces to

$$K_{\alpha;\beta\gamma} + K_{\gamma;\alpha\beta} + K_{\beta;\gamma\alpha} = 0, \tag{8.21}$$

and again from (8.18) this yields

$$K_{\gamma;\alpha\beta} = K_{\beta;\alpha\gamma} - K_{\beta;\gamma\alpha} = R^\rho{}_{\beta\alpha\gamma} K_\rho \tag{8.22}$$

Thus, in a given Riemannian manifold (where $g_{\alpha\beta}$ and, consequently, $R^\rho{}_{\alpha\beta\gamma}$ are given as functions of $\{x\}$ on open neighbourhoods of any nonsingular point), eq. (8.22) allows us to calculate **algebraically** $K_{\gamma;\alpha\beta}(p_0)$ if $K_\gamma(p_0)$ is given. If $K_{\gamma;\delta}(p_0)$ is given as well, then from the derivative of eq. (8.22) we can **algebraically** calculate $K_{\gamma;\alpha\beta\epsilon}(p_0)$. By differentiating (8.22) consecutively, we can then calculate all covariant derivatives of K_γ at p_0 and express them as functions of $K_\gamma(p_0)$ and $K_{\gamma;\delta}(p_0)$. Further, having all these derivatives (and so, equivalently, all partial derivatives of K_γ at p_0), we can calculate $K_\gamma(p)$ where $p \in M_n$ lies in such a neighbourhood of p_0 in which the Taylor series for $K_\alpha(p)$ is convergent. However, after each differentiation a new derivative of the Riemann tensor appears, so, in order that the series is convergent, $R^\rho{}_{\alpha\beta\gamma}$ must be analytic in that neighbourhood.[2] From this

[9] H. Stephani, *General Relativity.* Second edition. Cambridge University Press, Cambridge 1990.

[10] A. Krasiński, in *10th International Conference on General Relativity and Gravitation.* Abstracts of contributed papers. Edited by F. de Felice and A. Pascolini. University of Padua 1983, p. 290.

[2] In truth, the curvature tensor does not have to be analytic, but to see this one must use a different method of proof. See [11] J. Plebański and A. Krasiński, An introduction to general relativity and cosmology. Cambridge University Press 2006.

argument we see that $K_\gamma(p_0)$ and $K_{\gamma;\delta}(p_0)$ at any chosen $p_0 \in M$ are the data which are needed to determine $K_\gamma(p)$ uniquely (if $R^\rho{}_{\alpha\beta\gamma}(p_0)$ are not sufficiently differentiable, then simply another initial point is needed, not more data). But $K_{\gamma;\delta}$ obey (8.18), so $K_{\gamma;\delta}(p_0)$ are $\frac{1}{2}n(n-1)$ constants, and $K_\gamma(p_0)$ are n constants in an n-dimensional manifold. Thus, the Taylor series for $K_\gamma(p)$ will contain at most $\frac{1}{2}n(n+1)$ arbitrary constants multiplying various functions of $\{x\}$. The multipliers of those constants will be the basis solutions, and hence their number cannot exceed $\frac{1}{2}n(n+1)$.□

The prescription for finding the basis of the Killing vector fields for a given metric tensor is therefore the following:

1. Solve the Killing equations. The general solution will then depend on $N \leq \frac{1}{2}n(n+1)$ arbitrary constants, $k^\mu = K^\mu(A_1, \ldots, A_N, \{x\})$.

2. Calculate the basis

$$\underset{(i)}{k^\mu} \overset{\text{def}}{=} \frac{\partial K^\mu}{\partial A_i}, \qquad i = 1, \ldots, N. \tag{8.23}$$

Each $\underset{(i)}{k^\mu}$ generates a one-parameter subgroup of symmetries discussed in Sec. 8.2.

A possible confusion has to be explained here. For brevity, we say "Killing vectors", but in truth these are *vector fields*, whose components are functions. Hence, the number of linearly independent Killing vector *fields* can be larger than the dimension of the manifold. For example, in a flat Riemann space the number of linearly independent Killing vector fields is equal to the maximal one, $\frac{1}{2}n(n+1)$.

For tensor fields other than the metric tensor a finite basis may not exist, i. e. the general solution of the invariance equations will contain arbitrary functions rather than arbitrary constants; see Ref. [10] for examples.

8.5 Invariance of other tensor fields.

We investigated the conditions of invariance of the metric tensor in more detail because they are the most important and are most frequently met. Sometimes, though, we need to know the invariance transformations of other tensor fields. Repeating the reasoning (8.7) – (8.12) for the field of contravariant vectors, that is, assuming the condition $V'^\alpha(x) = V^\alpha(x)$ we would obtain the following equation:

$$k^\rho V^\alpha{}_{,\rho} - V^\rho k^\alpha{}_{,\rho} = 0, \tag{8.24}$$

where k^α is the generator of the transformation group.

Invariance conditions for other tensor fields are given in the exercises.

8.6 The Lie derivative.

If we trace the procedure that led to the Killing equations (8.12), and also the procedures leading to the other equations listed in the exercises, then we will notice that the invariance equations are obtained in the following steps:

1. Take the tensor field T (arbitrary indices) to be $T(t_0)$, where t_0 is the value of the orbit parameter corresponding to the identity transformation.

2. Using the transformation law for T under coordinate changes calculate $T(t)$ – the value of T transported to another point along the group orbit. The calculation is done up to terms linear in $t - t_0$ (the remaining terms are not neglected, but left in implicit form).

3. Calculate the quantity

$$-\lim_{t\to t_0} \frac{T(t) - T(t_0)}{t - t_0} \stackrel{\text{def}}{=} \underset{k}{\pounds} T, \tag{8.25}$$

where $k^\alpha = \frac{\mathrm{d}x'^\alpha}{\mathrm{d}t}\Big|_{t=t_0}$, and equate the result to zero.

The quantity $\left(-\underset{k}{\pounds} T\right)$ defined in (8.25) is thus the derivative of the tensor field T by the parameter of the transformation. As seen from (8.25), that derivative measures the speed of changes of the field T when it is transported along the orbits tangent to k^α. If $\underset{k}{\pounds} T = 0$, then the field transported along the orbit everywhere coincides with the tensor T defined before the transformation. The quantity $\underset{k}{\pounds} T$ is called the **Lie derivative**,[3] of the tensor field T along the vector field k. We have thus:

$$\left(\underset{k}{\pounds} T = 0\right) \Longleftrightarrow \left(T'(p) \equiv T(p)\right). \tag{8.26}$$

The Lie derivative has all the algebraic properties of ordinary differentiation: it is linear with respect to addition, gives zero when acting on a constant, and when acting on a tensor product it obeys the Leibniz rule:

$$\underset{k}{\pounds}(T_1 \otimes T_2) = \left(\underset{k}{\pounds} T_1\right) \otimes T_2 + T_1 \otimes \left(\underset{k}{\pounds} T_2\right). \tag{8.27}$$

These properties allow us to derive the formula for the Lie derivative of an arbitrary tensor field $T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}$ along any vector field k^α using (8.24), (8.52) and (8.53):

$$\underset{k}{\pounds} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} = k^\rho T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l,\rho} - \sum_{i=1}^{k} k^{\alpha_i}{}_{,\rho_i} T^{\alpha_1\ldots\rho_i\ldots\alpha_k}_{\beta_1\ldots\beta_l} + \sum_{j=1}^{k} k^{\rho_j}{}_{,\beta_j} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\rho_j\ldots\beta_l}, \tag{8.28}$$

where the sums extend over all positions of ρ_i and ρ_j in the series of indices. Thus, the Lie derivative acts similarly to the directional covariant derivative along k, the factors $(-k^\mu{}_{,\nu})$

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[3] The notion of the Lie derivative was introduced by the Polish mathematician Władysław Ślebodziński in 1931

[12] W. Ślebodziński, *Bulletins de la Classe des Sciences, Acad. Royale de Belg.* (5) **17**, 864 (1931). Reprinted in *Gen. Relativ. Gravit.* **42**, 2529 (2010), with an editorial note by A. Trautman, *Gen. Relativ. Gravit.* **42**, 2525 (2010) and author's biography by W. Roter, *Gen. Relativ. Gravit.* **42**, 2527 (2010).

while the term "Lie derivative" was proposed by van Dantzig and made popular by Schouten

[13] J. A. Schouten and E. R. van Kampen, *Prace Matematyczno-Fizyczne* **41**, 1 (1934). [**Note:** In citations of this paper, even by Schouten himself, the title of the journal and the date of publication are distorted. The citation here is correct.]

[14] J. A. Schouten and D. J. Struik, *Einführung in die neueren Methoden der Differentialgeometrie*, Band 1. P. Noordhoff N. V., Groningen - Batavia 1935, p. 142.

playing the role of the Christoffel symbols projected onto k, $-k^\mu{}_{,\nu} \longleftrightarrow \left\{ {\mu \atop \nu\rho} \right\} k^\rho$. One can now verify that for any tensor field T and any vector fields k and l we have

$$\underset{[k,l]}{£} T \equiv [\underset{k}{£}, \underset{l}{£}] T \stackrel{\text{def}}{=} \underset{k}{£}\left(\underset{l}{£} T\right) - \underset{l}{£}\left(\underset{k}{£} T\right), \tag{8.29}$$

and so $[k, l]$ generates an invariance of T if k and l do. (**Note:** Verifying (8.29) by direct application of (8.28) leads through very large intermediate expressions. A simpler way is to use the same method that we used to derive the Ricci formula (6.10): (1) verify that for any tensors T_1 and T_2 $[\underset{k}{£}, \underset{l}{£}]\,(T_1 \otimes T_2) = \left\{[\underset{k}{£}, \underset{l}{£}]\,(T_1)\right\} \otimes T_2 + T_1 \otimes \left\{[\underset{k}{£}, \underset{l}{£}]\,(T_2)\right\}$; (2) observe that the left-hand sides of (8.52), (8.53) and (8.24) are tensors and verify directly, using these equations, that (8.29) holds for scalar, contravariant and covariant vector fields, and finally (3) use (4.16) to represent the tensor as a multi-linear combination of contra- and covariant vector fields with scalar coefficients. Since (8.29) holds for each factor in each term of the sum, it holds for the whole sum.)

8.7 The algebra of Killing vector fields.

Since for the proper Killing fields a finite basis exists, we conclude from the last statement of the previous section that there exist such constants $C^l{}_{ij}$ that, for the basis fields

$$[\underset{(i)}{k}, \underset{(j)}{k}]^\alpha = C^l{}_{ij} \underset{(l)}{k}{}^\alpha \qquad \text{(sum over } l\text{)}. \tag{8.30}$$

The constants $C^l{}_{ij}$ are called **structure constants** of the symmetry group. For generators of invariances of other tensor fields, the coefficients $C^l{}_{ij}$ will not necessarily be constant.

8.8 Spherically symmetric 4-dimensional Riemann spaces.

A Riemann space is spherically symmetric when the group of rotations around a point, $O(3)$, is its isometry group. Its metric tensor must thus obey the Killing equations for each generator of $O(3)$. We shall first derive the formulae for these generators.

The orbits of $O(3)$ are two-dimensional spheres. Each sphere of radius R can be embedded in a 3-dimensional Euclidean space E^3. Its equation is

$$x^2 + y^2 + z^2 = R^2, \tag{8.31}$$

where x, y, z are Cartesian coordinates in E^3. The rotation around the centre of the sphere by the angle α in the plane (x^i, x^j) is then

$$\begin{aligned} x'^i &= x^i \cos\alpha + x^j \sin\alpha, \\ x'^j &= -x^i \sin\alpha + x^j \cos\alpha, \end{aligned}$$

$$x'^k \;= x^k \qquad \text{for } i \neq k \neq j. \tag{8.32}$$

The angle α is here the group parameter. From (8.6), the corresponding Killing vector is

$$\underset{[i,j]}{k^\mu} = x^j \delta^\mu{}_i - x^i \delta^\mu{}_j. \tag{8.33}$$

An arbitrary transformation of the sphere into itself can be described as a composition of three consecutive rotations around different axes. Hence, a basis of the space of Killing vectors will be three generators corresponding to rotations around three different axes.

It is often convenient to represent the Killing vectors by the corresponding operators of directional derivatives, also called generators

$$\underset{(i)}{J} \overset{\text{def}}{=} -\underset{(i)}{k^\mu} \frac{\partial}{\partial x^\mu}. \tag{8.34}$$

We will choose as our basis of Killing fields the generators of rotations around the three axes of the rectangular Cartesian coordinate system. They are

$$\underset{[xy]}{J} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad \underset{[yz]}{J} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad \underset{[xz]}{J} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}. \tag{8.35}$$

Now let us transform the generators to the spherical coordinates

$$x = r\sin\vartheta\cos\varphi, \qquad y = r\sin\vartheta\sin\varphi, \qquad z = r\cos\vartheta. \tag{8.36}$$

In these coordinates the generators become, up to sign,

$$\begin{aligned} \underset{[xy]}{J} &= \frac{\partial}{\partial\varphi}, \qquad \underset{[yz]}{J} = \sin\varphi\frac{\partial}{\partial\vartheta} + \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}, \\ \underset{[xz]}{J} &= \cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}. \end{aligned} \tag{8.37}$$

Since the coordinates ϑ and φ are defined inside the spheres, we can use them as coordinates in the whole Riemann space. Let us denote the two remaining coordinates t and r. We shall now solve the Killing equations for the metric tensor $g_{\alpha\beta}(t, r, \vartheta, \varphi)$, where $x^0 = t$, $x^1 = r$, $x^2 = \vartheta$, $x^3 = \varphi$, with the Killing vectors given by (8.37), thus

$$\begin{aligned} \underset{(1)}{k^\alpha} &= \delta^\alpha{}_3, \qquad \underset{(2)}{k^\alpha} = \sin\varphi\delta^\alpha{}_2 + \cos\varphi\cot\vartheta\delta^\alpha{}_3, \\ \underset{(3)}{k^\alpha} &= \cos\varphi\delta^\alpha{}_2 - \sin\varphi\cot\vartheta\delta^\alpha{}_3. \end{aligned} \tag{8.38}$$

The Killing equations for the vector $\underset{(1)}{k^\alpha}$ reduce to $g_{\alpha\beta,3} = 0$, that is, the whole metric tensor is independent of $x^3 = \varphi$. For $\underset{(2)}{k^\alpha}$ and $\underset{(3)}{k^\alpha}$ the Killing equations are

$$\begin{aligned} \sin\varphi\frac{\partial}{\partial\vartheta}g_{\alpha\beta} \quad &+(\sin\varphi)_{,\alpha}\, g_{2\beta} + (\sin\varphi)_{,\beta}\, g_{\alpha 2} \\ &+(\cos\varphi\cot\vartheta)_{,\alpha}\, g_{3\beta} + (\cos\varphi\cot\vartheta)_{,\beta}\, g_{\alpha 3} = 0, \end{aligned} \tag{8.39}$$

$$\cos\varphi \frac{\partial}{\partial\vartheta} g_{\alpha\beta} \quad +(\cos\varphi)_{,\alpha}\, g_{2\beta} + (\cos\varphi)_{,\beta}\, g_{\alpha 2} \\ -(\sin\varphi\cot\vartheta)_{,\alpha}\, g_{3\beta} - (\sin\varphi\cot\vartheta)_{,\beta}\, g_{\alpha 3} = 0. \tag{8.40}$$

To simplify further calculations, we will replace (8.39) – (8.40) by two combinations of them. Multiply (8.39) by $\cos\varphi$, (8.40) by $\sin\varphi$ and subtract the results. Using the identities

$$\begin{aligned} \sin\varphi(\sin\varphi)_{,\alpha} + \cos\varphi(\cos\varphi)_{,\alpha} &\equiv 0, \\ \cos\varphi(\sin\varphi)_{,\alpha} - \sin\varphi(\cos\varphi)_{,\alpha} &\equiv \varphi_{,\alpha} \end{aligned} \tag{8.41}$$

we obtain

$$\varphi_{,\alpha}\, g_{2\beta} + \varphi_{,\beta}\, g_{\alpha 2} + (\cot\vartheta)_{,\alpha}\, g_{3\beta} + (\cot\vartheta)_{,\beta}\, g_{\alpha 3} = 0. \tag{8.42}$$

Now let us multiply (8.39) by $\sin\varphi$, (8.40) by $\cos\varphi$, and add the results. Using (8.41) again we get

$$\frac{\partial}{\partial\vartheta} g_{\alpha\beta} - \varphi_{,\alpha}\cot\vartheta g_{3\beta} - \varphi_{,\beta}\cot\vartheta g_{\alpha 3} = 0. \tag{8.43}$$

Equation (8.42) is algebraic. Taking the various values of α and β we get from it:

- For $2 \neq \alpha \neq 3$, $2 \neq \beta \neq 3$ the equation is fulfilled identically.
- For $2 \neq \alpha \neq 3$, $\beta = 2$:
$$g_{\alpha 3} = 0, \qquad \alpha = 0, 1. \tag{8.44}$$
- For $2 \neq \alpha \neq 3$, $\beta = 3$:
$$g_{\alpha 2} = 0, \qquad \alpha = 0, 1. \tag{8.45}$$
- For $\alpha = 2$, $\beta = 2$:
$$g_{23} + g_{32} \equiv 2g_{23} = 0. \tag{8.46}$$
- For $\alpha = 2$, $\beta = 3$: $g_{22} - g_{33}/\sin^2\vartheta = 0$, which means
$$g_{33} = g_{22}\sin^2\vartheta. \tag{8.47}$$
- For $\alpha = \beta = 3$ eq. (8.47) follows once more.

Now we take the same cases for eq. (8.43) and obtain:

- For $2 \neq \alpha \neq 3$, $2 \neq \beta \neq 3$:
$$\frac{\partial}{\partial\vartheta} g_{\alpha\beta} = 0, \qquad \alpha, \beta = 0, 1. \tag{8.48}$$
- For $2 \neq \alpha \neq 3$, $\beta = 2$ the result is an identity in consequence of (8.45).
- For $2 \neq \alpha \neq 3$, $\beta = 3$ the result is an identity in consequence of (8.44).
- For $\alpha = 2$, $\beta = 2$:
$$\frac{\partial}{\partial\vartheta} g_{22} = 0. \tag{8.49}$$

- For $\alpha = 2$, $\beta = 3$ the result is an identity in consequence of (8.46).
- For $\alpha = \beta = 3$ the result is $\dfrac{\partial}{\partial\vartheta} g_{33} - 2\cot\vartheta g_{33} = 0$, which is an identity in consequence of (8.47).

Hence, finally, eqs. (8.42) – (8.43) determined the following metric

$$\mathrm{d}s^2 = \alpha(t,r)\mathrm{d}t^2 + 2\beta(t,r)\mathrm{d}t\mathrm{d}r + \gamma(t,r)\mathrm{d}r^2 + \delta(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right). \tag{8.50}$$

This is the general 4-dimensional spherically symmetric metric. Note that, in consequence of the assumption that the 2-spheres are subspaces of the Riemann space (they are the orbits of the symmetry group), we have obtained a limitation on the signature: the signs at $\mathrm{d}\vartheta^2$ and at $\mathrm{d}\varphi^2$ must be the same.

We have not assumed anything about the subspaces of the variables (t, r). Hence, arbitrary nonsingular coordinate transformations can be carried out within those subspaces:

$$t = f(t', r'), \qquad r = g(t', r'), \tag{8.51}$$

where f and g are arbitrary functions subject to the condition $\partial(t,r)/\partial(t',r') \neq 0$. After such a transformation the function $\delta(t,r)$ will preserve its value, while α, β and γ will change to combinations of α, β and γ that will still depend only on t' and r'.

The orbits of the group $O(3)$ are the subspaces $\{t = \text{const}, r = \text{const}\}$ whose metric form, in the coordinates of (8.50), is ${\mathrm{d}s_2}^2 = \delta\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)$ (in these subspaces δ is constant). The centre of symmetry is where $\delta(t,r) = 0$. However, there is no guarantee that such a point exists within the manifold. For example, if δ is a constant in the whole Riemann space (which is a property invariant under (8.51)), then the centre of symmetry does not exist. This is an analogy to a cylinder or a one-sheeted hyperboloid: these surfaces are rotationally symmetric, but no point on the surface is the centre of rotation.

If the functions α, β, γ and δ in (8.50) are independent of r, then there exists a fourth Killing field $\underset{(4)}{k}{}^{\alpha} = {\delta^{\alpha}}_1$. The symmetry group of such spacetimes is called the **Kantowski – Sachs symmetry** [15]. In these spacetimes those hypersurfaces $t =$ constant for which $\delta(t) \neq 0$ have no center of symmetry. We will come back to them in Chapter 9.

8.9 Exercises

1. Find the coordinate transformation corresponding to the field of generators $k^{\alpha} = {\delta^{\alpha}}_{\alpha_0}$, where α_0 is the label of one of the coordinates.

2. Find the coordinate transformation corresponding to the field of generators $k^{\mu} = x^i {\delta^{\mu}}_j - x^j {\delta^{\mu}}_i$, where i and j are the labels of fixed coordinates. Show that when (x^i, x^j) are Cartesian coordinates, the transformation found here is the rotation in the plane of (x^i, x^j).

[15] R. Kantowski and R. K. Sachs, *J. Math. Phys.* **7**, 443 (1966).

3. Solve the Killing equations for the field of Killing vectors from exercise 1.

4. Show that if the parameter λ of the integral lines of a Killing field $k^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$ is chosen as a coordinate in the Riemann space, then a tensor invariant under the transformations generated by k^α is simply independent of λ.

5. Show that if there exist at least two linearly independent generators k^α and l^α connected with invariances of a tensor field $T_{\alpha\beta}$, then the corresponding orbit parameters λ and τ can be chosen as coordinates on M_n if and only if $[k,l]^\alpha \stackrel{\text{def}}{=} k^\rho l^\alpha{}_{,\rho} - l^\rho k^\alpha{}_{,\rho} = 0$. In that case $\partial T_{\alpha\beta}/\partial t = \partial T_{\alpha\beta}/\partial \tau = 0$.

Note: This is in fact just a different wording of Exercise 2 to chapter 6.

6. Prove that if k^α and l^α are Killing fields in a certain Riemann space, then so is their commutator $[k,l]^\alpha$. (**Note:** This result is nontrivial only when $[k,l]^\alpha \neq 0$.)

7. Show that the condition of invariance of a scalar field ϕ with respect to the transformation group generated by the vector field k^α is

$$k^\alpha \phi_{,\alpha} = 0. \tag{8.52}$$

8. Show that the condition of invariance of a covariant vector field ω_α with respect to the transformation group generated by the vector field k^α is

$$k^\rho \omega_{\alpha,\rho} + k^\rho{}_{,\alpha}\, \omega_\rho = 0. \tag{8.53}$$

9. Find and interpret all the Killing fields for the Minkowski spacetime in the Cartesian coordinates, in which $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$. Find the corresponding isometries. Identify the isometries that should be known to you from the special relativity course: the special Lorentz transformations along the $x-$, $y-$ and $z-$directions, and the rotations in the planes $\{x,y\}$, $\{y,z\}$ and $\{x,z\}$. Verify that they are isometries indeed (i.e. substitute these transformations into the metric form and see what happens). Calculate all the structure constants of the full group.

10. One of the coordinate representations of 4-dimensional spaces of constant curvature with the signature $(+---)$ (the de Sitter spacetimes [16]) is

$$\mathrm{d}s^2 = \left(1 - \Lambda r^2\right)\mathrm{d}t^2 - \frac{1}{1 - \Lambda r^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right), \tag{8.54}$$

where Λ is an arbitrary constant. Find all the Killing fields for this metric, in each of the cases $\Lambda > 0$, $\Lambda < 0$ and $\Lambda = 0$ (the last case is just the Minkowski spacetime in spherical coordinates). Find the structure constants of these groups. Take the limit $\Lambda \to 0$ of the first and second case and see what happens with the structure constants.

Hint: You may prefer to find the Killing fields for the case $\Lambda = 0$ by transforming the results of Exercise 9 to spherical coordinates.

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[16] W. de Sitter, *Mon. Not. Roy. Astr. Soc.* **78**, 3 (1917).

Chapter 9

The spatially homogeneous Bianchi-type spacetimes.

9.1 The Bianchi classification of 3-dimensional Lie algebras.

For reasons to be explained further, the case when the symmetry group of a spacetime is 3-dimensional is important in relativity. The aim of the Bianchi classification is to sort out the 3-dimensional groups which are inequivalent in the following sense.

A basis of vector fields is not defined uniquely. If $\underset{(i)}{k}$ is a basis, then

$$\underset{(i')}{\widetilde{k}}{}^{\alpha} \overset{\text{def}}{=} A_{i'}{}^{j} \underset{(j)}{k}{}^{\alpha} \qquad \text{(sum over } j\text{)} \tag{9.1}$$

is also a basis, provided the matrix $A_{i'}{}^{j}$ (of constant elements) is nonsingular. Such a change of basis is accompanied by the following change of the structure constants

$$\widetilde{C}^{l'}_{i'j'} = \left(A^{-1}\right)^{l'}{}_{l} A_{i'}{}^{i} A_{j'}{}^{j} C^{l}{}_{ij}. \tag{9.2}$$

Thus, two sets of structure constants that are related by (9.2) correspond to two bases that generate isomorphic groups. How can one recognise whether any matrix A obeying (9.2) exists for two given sets of $C^{l}{}_{ij}$? The answer is provided by the Bianchi classification.

The method of presentation used here was introduced by Engelbert Schücking in a seminar talk at the Hamburg University in the 1950-s and diffused into public knowledge via notes taken by Wolfgang Kundt [17], see Ref. [18] for a description of the story. The earliest papers that introduced this approach to the literature were those by Estabrook, Wahlquist and Behr [19] and by Ellis and MacCallum [20, 21]. The classification was

[17] W. Kundt, *Gen. Relativ. Gravit.* **35**, 491 (2003).
[18] A. Krasiński, C. G. Behr, E. Schücking, F. B. Estabrook, H. D. Wahlquist, G. F. R. Ellis, R. Jantzen and W. Kundt, *Gen. Relativ. Gravit.* **35**, 475 (2003).
[19] F. B. Estabrook, H. D. Wahlquist and C. G. Behr, *J. Math. Phys.* **9**, 497 (1968).
[20] G. F. R. Ellis and M. A. H. MacCallum, *Commun. Math. Phys.* **12**, 108 (1969).
[21] G. F. R. Ellis and M. A. H. MacCallum, *Commun. Math. Phys.* **19**, 31 (1970).

originally introduced by Bianchi [22], but his method requires a long presentation.[4]

From (8.30) it follows that ${C^l}_{ij} = -{C^l}_{ji}$, so for an N-dimensional group the number of structure constants cannot exceed $N \times \frac{1}{2}N(N-1)$. But through the Jacobi identity

$$[[\underset{(i)}{k}, \underset{(j)}{k}], \underset{(l)}{k}] + [[\underset{(j)}{k}, \underset{(l)}{k}], \underset{(i)}{k}] + [[\underset{(l)}{k}, \underset{(i)}{k}], \underset{(j)}{k}] = 0, \tag{9.3}$$

the definition (8.30) implies a further limitation on ${C^l}_{ij}$:

$${C^m}_{ij}{C^n}_{ml} + {C^m}_{jl}{C^n}_{mi} + {C^m}_{li}{C^n}_{mj} = 0. \tag{9.4}$$

For $N = 3$, $\frac{1}{2}N^2(N-1) = 9$ and equals the number of elements of a 3×3 matrix. Thus for a 3-dimensional group all structure constants can be packed into a single 3×3 matrix, let us call it H^{ab}. The 1–1 correspondence between ${C^l}_{ij}$ and the elements of H^{ab} is

$$H^{ab} = \frac{1}{2}\epsilon^{akl}{C^b}_{kl} \Longleftrightarrow {C^i}_{jk} = \epsilon_{sjk}H^{si}, \tag{9.5}$$

where ϵ^{akl} and ϵ_{sjk} are the Levi-Civita symbols. The matrix H^{ab} can now be split into the symmetric part

$$n^{ab} \overset{\text{def}}{=} H^{(ab)} = \frac{1}{2}\left(H^{ab} + H^{ba}\right) \tag{9.6}$$

and the antisymmetric part $H^{[ab]}$. But in 3 dimensions there is a 1–1 correspondence between antisymmetric matrices and vectors, so $H^{[ab]}$ can be represented by

$$a_i \overset{\text{def}}{=} -\frac{1}{2}\epsilon_{ijk}H^{[jk]} \Longleftrightarrow H^{[ij]} = -\epsilon^{ijm}a_m. \tag{9.7}$$

From (9.5), (9.6) and (9.7) we have

$${C^l}_{jk} = \epsilon_{sjk}n^{sl} - \delta^{lm}_{jk}a_m. \tag{9.8}$$

For $N = 3$ (9.4) may be written as ${C^m}_{ij}{C^n}_{ml}\epsilon^{ijl} = 0$, and with use of (9.8) this implies

$$n^{is}a_s = 0, \tag{9.9}$$

[22] L. Bianchi, *Memorie di Matematica e di Fisica della Societa Itialiana delle Scienze* **11**, 267 (1898). English translation: *Gen. Relativ. Gravit.* **33**, 2171 (2002), with an editorial note by R. Jantzen, *Gen. Relativ. Gravit.* **33**, 2157 (2001) and author's biography by R. Jantzen, *Gen. Relativ. Gravit.* **33**, 2168 (2001).

[4] Note that the Bianchi paper was published 17 years before the paper by Einstein that introduced relativity. Bianchi considered his classification from the point of view of the theory of Lie algebras, with no relation to relativity. Its importance for relativity was recognised much later. The first paper that explicitly introduced the Bianchi classes into relativity was that by Taub

[23] A. H. Taub *Ann. Math.*, **53**, 472 (1951). Reprinted in *Gen. Relativ. Gravit.* **36**, 2699 (2004), with an editorial note by M. A. H. MacCallum, *Gen. Relativ. Gravit.* **36**, 2689 (2004) and author's biography by B. Mashhoon, *Gen. Relativ. Gravit.* **36**, 2695 (2004).

Taub's inspiration is said to have come from Gödel

[24] K. Gödel, *Rev. Mod. Phys.* **21**, 447 (1949). Reprinted in *Gen. Relativ. Gravit.* **32**, 1409 (2000), with an editorial note by G. F. R. Ellis, *Gen. Relativ. Gravit.* **32**, 1399 (2000), and author's biography by A. Krasiński, *Gen. Relativ. Gravit.* **32**, 1407 (2000). See the note by Jantzen

[25] R. Jantzen, *Gen. Relativ. Gravit.* **33**, 2157 (2001)

for more on this story. Bianchi sorted out the different types by the properties of the derived algebras.

i. e. either $a_i = 0$ or else n^{is} has at least one zero eigenvalue.

From (9.2) and (9.5) it follows that n^{ab} transforms, under the change of basis (9.1), by

$$\tilde{n}^{i'j'} = (\det A)^{-1} \left(A^{-1}\right)^{i'}{}_{i} \left(A^{-1}\right)^{j'}{}_{j} n^{ij}, \tag{9.10}$$

while a_i transforms by $\tilde{a}_{i'} = A_{i'}{}^{i} a_i$. The transformations (9.10) can be used to diagonalise n^{ij}. Let us then assume that the basis $\underset{(i)}{k}$ was chosen so that n^{ij} is of the form

$$n^{ij} = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix}. \tag{9.11}$$

Transformations which permute (n_1, n_2, n_3) are still allowed. Therefore if any of the n_i is zero, it can be moved to the upper left corner in (9.11), i. e. we can assume that

$$n_1 = 0 \qquad \text{if} \qquad a_i \neq 0. \tag{9.12}$$

If $n_1 = 0$ is the single zero eigenvalue, then in view of (9.9) the vector a will assume, after such a choice of basis, the form

$$a_i = [a, 0, 0]. \tag{9.13}$$

If $n_1 = 0$ is a multiple eigenvalue, then we still have the freedom to rotate a within the eigenspace of n_1. We can then rotate a so that it will obey (9.13). Consequently, (9.13) may be assumed always (this covers the case $a_i = 0$). In such a basis, (9.9) reduces to

$$a n_1 = 0. \tag{9.14}$$

Using all the information about $C^l{}_{ij}$, the commutators become:

$$[\underset{(1)}{k}, \underset{(2)}{k}] = a \underset{(2)}{k} + n_3 \underset{(3)}{k}, \qquad [\underset{(2)}{k}, \underset{(3)}{k}] = n_1 \underset{(1)}{k}, \tag{9.15}$$

$$[\underset{(3)}{k}, \underset{(1)}{k}] = n_2 \underset{(2)}{k} - a \underset{(3)}{k}. \tag{9.16}$$

So far we used only rotations of $\underset{(i)}{k}$. We may still scale $\underset{(i)}{k}$ without changing their directions, by $\underset{(i)}{k} = C_i \underset{(i)}{k'}$ (no sum). After such scalings, (9.15) – (9.16) change to

$$[\underset{(1)}{k}, \underset{(2)}{k}] = \frac{a}{C_1} \underset{(2)}{k} + \frac{C_3}{C_1 C_2} n_3 \underset{(3)}{k}, \qquad [\underset{(2)}{k}, \underset{(3)}{k}] = \frac{C_1}{C_2 C_3} n_1 \underset{(1)}{k}, \tag{9.17}$$

$$[\underset{(3)}{k}, \underset{(1)}{k}] = \frac{C_2}{C_1 C_3} n_2 \underset{(2)}{k} - \frac{a}{C_1} \underset{(3)}{k} \tag{9.18}$$

(primes were dropped). We now want to use the scalings to make a, n_1, n_2, n_3 as simple as possible. By choosing C_1, C_2, C_3 we can scale those of (a, n_1, n_2, n_3) that are nonzero. A nonzero value can never be made zero thereby. Hence, the preliminary classification into different cases is as shown in Table 9.1, where S stands for "something nonzero".

However, not all the entries in Table 9.1 must be considered separately. Permutations of the basis vectors that do not violate (9.14) are still allowed. Clearly, any permutation of

Table 9.1: A preliminary Bianchi classification

a	0	0	0	0	0	0	0	0	S	S	S	S
n_1	0	0	0	0	S	S	S	S	0	0	0	0
n_2	0	0	S	S	0	0	S	S	0	0	S	S
n_3	0	S	0	S	0	S	0	S	0	S	0	S
CASE NUMBER	1	/	/	/	2	/	3	4	5	6	/	7

$\{n_1, n_2, n_3\}$ is allowed when $a = 0$, and with $a \neq 0$ we can still permute n_2 and n_3. Hence, only the cases indicated in the last line of the table have a chance to be inequivalent.

We will see that the classification of the algebras into inequivalent types does not match the columns of Table 9.1 – the table will only serve to make the presentation orderly. The labels for the types come from Bianchi [22]. By tradition, his numbering is still in use, although it does not seem natural in the derivation presented below.

Let us consider the consecutive columns of Table 9.1.

(1) $a = n_1 = n_2 = n_3 = 0$.

This is Bianchi type I where all commutators are zero.

(2) $a = n_2 = n_3 = 0$, $n_1 \neq 0$.

Taking $C_1 = C_2C_3/n_1$ we obtain $n'_1 = 1$. This is Bianchi type II.

(3) $a = n_3 = 0$, $n_1 \neq 0 \neq n_2$.

Taking $C_1 = C_2C_3/n_1$ we obtain $n'_1 = 1$. However, as seen from (9.18), we have then $n'_2 = n_1n_2/{C_3}^2$ and by choosing C_3 we will not be able to change the sign of n'_2. Therefore we must consider two cases separately:

(3a) $n_1n_2 > 0$.

Then we take $C_3 = (n_1n_2)^{1/2}$ and obtain $n'_2 = 1$. This is a subcase of Bianchi's type VII. Bianchi himself called it type VII_1, today it is called type VII_0.

(3b) $n_1n_2 < 0$.

Then we take $C_3 = (-n_1n_2)^{1/2}$ and obtain $n'_2 = -1$. This is a subcase of Bianchi's type VI, called today VI_0. Bianchi noted that this case requires a separate treatment, but the final result fits well within the general type VI, so he did not give it any special name.

(4) $a = 0$, $n_1n_2n_3 \neq 0$.

We take $C_1 = C_2C_3/n_1$ and obtain $n'_1 = 1$. But then, as before, $n'_2 = n_1n_2/{C_3}^2$, and the two possible signs of n_1n_2 have to be considered separately.

(4a) $n_1n_2 > 0$.

We take $C_3 = (n_1n_2)^{1/2}$ and obtain $n'_2 = 1$. However, $n'_3 = n_1n_3/{C_2}^2$, and two further subcases arise:

($4\mathrm{a}_1$) $n_1n_3 > 0$.

Then we take $C_2 = (n_1 n_3)^{1/2}$ and obtain $n'_3 = 1$. Thus finally $n'_1 = n'_2 = n'_3 = 1$, $a = 0$. This is Bianchi type IX.

(4a$_2$) $n_1 n_3 < 0$.

Then we take $C_2 = (-n_1 n_3)^{1/2}$ and obtain $n'_3 = -1$. Hence finally $n'_1 = n'_2 = 1 = -n'_3$. This is Bianchi type VIII.

(4b) $n_1 n_2 < 0$.

We then take $C_3 = (-n_1 n_2)^{1/2}$ obtaining $n'_2 = -1$. But then we must again consider the same two subcases as before for C_2:

(4b$_1$) $n_1 n_3 > 0$.

Then we take $C_2 = (n_1 n_3)^{1/2}$ and obtain $n'_3 = 1$. Through the basis change (still allowed!)

$$\underset{(3)}{k} = \underset{(2)}{\widetilde{k}}, \qquad \underset{(2)}{k} = -\underset{(3)}{\widetilde{k}}$$

we then obtain the same parameter values as in case (4a$_2$).

(4b$_2$) $n_1 n_3 < 0$.

Then we take $C_2 = (-n_1 n_3)^{1/2}$ obtaining $n'_3 = -1$. Again through the basis change:

$$\underset{(1)}{k} = -\underset{(3)}{\widetilde{k}}, \qquad \underset{(3)}{k} = -\underset{(2)}{\widetilde{k}}, \qquad \underset{(2)}{k} = -\underset{(1)}{\widetilde{k}}$$

we arrive back at the case (4a$_2$).

(5) $a \neq 0$, $n_1 = n_2 = n_3 = 0$.

Taking $C_1 = a$ we obtain $a' = 1$. This is Bianchi type V.

(6) $a \neq 0 \neq n_3$, $n_1 = n_2 = 0$.

We take $C_1 = a$, $C_3 = aC_2/n_3$ and obtain $a' = 1 = n'_3$. This is Bianchi type IV.

(7) $a n_2 n_3 \neq 0$, $n_1 = 0$.

Taking $C_2 = C_1 C_3 / n_2$ we obtain $n'_2 = 1$. We lose again the possibility to change the sign of $n'_3 = n_2 n_3 / {C_1}^2$ and must consider two cases separately:

(7a) $n_2 n_3 > 0$.

Then we take $C_1 = (n_2 n_3)^{1/2}$ and obtain $n'_3 = 1$. Since however we fix C_1 in this way, we have no possibility left to scale a. Hence, with $n'_3 = 1$, the value of a remains arbitrary. Note that then the algebras corresponding to $n_2 = n_3 = 1$ but different values of *a are not* equivalent. Bianchi called this case type VII$_2$, but, as opposed to previous types, this is a one-parameter family of inequivalent types. With $a = 0$, an algebra equivalent to the one obtained in (3a) results. This is seen if, with $a = 0$, we carry out the basis change

$$\underset{(1)}{k} = \underset{(3)}{\widetilde{k}}, \qquad \underset{(3)}{k} = -\underset{(1)}{\widetilde{k}}$$

The general type VII, with $a \neq 0$, is denoted today VII$_h$.

(7b) $n_2 n_3 < 0$.

Table 9.2: The Bianchi classification

a	0	0	0	0	0	0	1	1	a	a	1
n_1	0	1	1	1	1	1	0	0	0	0	0
n_2	0	0	1	-1	1	1	0	0	1	1	1
n_3	0	0	0	0	1	-1	0	1	1	-1	-1
BIANCHI TYPE	I	II	VII_0	VI_0	IX	VIII	V	IV	VII_h	VI_h	III

Then we take $C_1 = (-n_2 n_3)^{1/2}$ and obtain $n_3' = -1$. Just as before, we are left then with no possibility to change a. Again, this is a one-parameter family of inequivalent types. Bianchi denoted it VI, today it is denoted VI_h. With $a = 0$, the algebra obtained in (3b) results, i. e. type VI_0. To see this, the same basis change as in (7a) is necessary.

In Bianchi's scheme the subcase of type VI corresponding to $a = 1$ emerged as a separate type which he called type III. In it:

$$a = n_2 = 1 = -n_3, \qquad n_1 = 0.$$

The final classification is shown in Table 9.2.

9.2 The dimension of the group vs. the dimension of the orbit.

In. Sec. 8.2 we defined the orbits of 1-parameter families of transformations. But the notion of an orbit can be defined also for a family (or group) of transformations that has several parameters; such orbits can be multidimensional curved spaces.

The Bianchi classification was done without any reference to the way in which the generators act on the manifold M_n. Two situations are possible:

1. The 3 generators (which are by definition linearly independent as *vector fields*) may also be linearly independent as *vectors* at each point of M_n. This happens e.g. for generators of translations in $\mathbb{R}^3$.

2. The 3 generators may be linearly dependent at each point of M_n. This happens e.g. for the generators of the group $O(3)$ which is 3-dimensional, but has 2-dimensional orbits.

A 3-dimensional isometry group cannot have 1-dimensional orbits (from the Killing equations: if the orbits are 1-dimensional, then any two generators will be proportional to each other with a constant factor).

If the orbits are 2-dimensional, then their curvature is characterised by one scalar that must be invariant under the action of the group, i. e. constant. The constant curvature may be positive (the orbits are then 2-dimensional spheres) or zero (the orbits are then 2-planes) or negative (such a surface has the metric $\mathrm{d}s^2 = (1 + a^2 r^2)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2$, but cannot be embedded in the $\mathbb{R}^3$ with a positive-definite Euclidean metric). Then, the 3 algebras of generators are of Bianchi types IX, VII_0 and VIII, respectively. Only these Bianchi types are possible for groups which have 2-dimensional orbits.

If a 3-dimensional group has 3-dimensional orbits, then the scalar curvature of the orbits must also be constant. However, in 3 dimensions the scalar curvature does not fully characterise the curvature of space, there exists also the Ricci tensor. Therefore more geometries are possible, they will be briefly discussed in Sec. 9.6.

9.3 Action of a group on a manifold.

If a 3-dimensional group of transformations of M_n into itself has 3-dimensional orbits, then several situations are possible, e. g.:

1. The mappings may or may not be symmetries of M_n.
2. The orbits may be timelike, spacelike or null hypersurfaces in M_n.

In each situation one can consider the various Bianchi types. Examples of spacetimes are known where the orbits are timelike (e. g. [26] for Bianchi type I and [27, 28] for more general types; in the latter papers there are examples of null orbits). A systematic investigation of all the spacetimes with 3-dimensional timelike orbits of the Bianchi groups was done by Harness [29]. Examples are also known where 3 dimensional groups act as groups of symmetries on certain preferred 3-dimensional submanifolds of M_n, but are not symmetries of the whole M_n (these are the so called spacetimes with intrinsic symmetries, the term was introduced by Collins [30], for examples see [31] and [32]). A brief discussion of general properties of spacetimes for which a 3-dimensional symmetry group has 3-dimensional null orbits is given in [33]. A systematic investigation was done of the case when the group has 3-dimensional spacelike orbits and the group transformations are conformal symmetries of the manifold, i.e., change $g_{\alpha\beta}$ into $\phi g_{\alpha\beta}$, with constant ϕ (these are called **self-similar** spacetimes), see Ref. [34] for the original reference.

Most effort went into investigating spacetimes in which the 3-dimensional orbits are spacelike and the group in question is a group of symmetries of M_n (but examples are known where the orbits are spacelike in one part of M_n and timelike elsewhere, see [35] and [36]). Before we consider these in more detail, a few definitions must be given.

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[26] A. Krasiński, *Acta. Phys. Polon.* **B5**, 411 (1974), and **B6**, 223 (1975).

[27] A. Krasiński, *J. Math. Phys.* **39**, 380 (1998); **39**, 401 (1998); **39**, 2148 (1998).

[28] A. Krasiński, *J. Math. Phys.* **42**, 355 and 3628 (2001).

[29] R. S. Harness, *J. Phys.* **A15**, 135 (1982).

[30] C. B. Collins, *Gen. Relativ. Gravit.* **10**, 925 (1979).

[31] A. Krasiński, *Gen. Relativ. Gravit.* **13**, 1021 (1981).

[32] T. Wolf, *Exakte Lösungen der Einsteinschen Feldgleichungen mit flachen Schnitten.* PhD Thesis, University of Jena 1985.

[33]. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations.* 2nd Edition. Cambridge University Press 2003, Sec. 24.2.

[34] D. M. Eardley, *Commun. Math. Phys.* **37**, 287 (1974).

[35] C. B. Collins and J. Wainwright, *Phys. Rev.* **D27**, 1209 (1983).

[36] C. B. Collins and G. F. R. Ellis, *Phys. Rep.* **56** no 2, 65 (1979).

9.4 Groups acting transitively, homogeneous spaces.

Let $\mathrm{Orb}(p, G)$ denote the orbit of the point $p \in M_n$ under the action of the group G. We say that G **acts transitively** on a manifold $S \subseteq M_n$ when for every $q \in S$, $\mathrm{Orb}(q, G) = S$. *Examples*: the group $O(3)$ acts transitively on the surface of a sphere, the group of arbitrary translations in $\mathbb{R}^n$ acts transitively on $\mathbb{R}^n$. The space S on which G acts transitively is called **homogeneous with respect to** G.

If, for every $q \in S$ (S assumed homogeneous with respect to G) there exists a subgroup $H \subset G$ such that $Hq = q$, then G is said to act **multiply transitively** on S. A group acting transitively, but not multiply, acts **simply transitively**. *Examples* : The group of arbitrary translations in $\mathbb{R}^n$ acts simply transitively on $\mathbb{R}^n$; the group $O(3)$ acts multiply transitively on a 2-sphere since each point p of the sphere remains unchanged by all rotations around the axis which passes through p.

A 4-dimensional manifold M_4 with the metric g of signature $(+ - --)$ is called a **spatially homogeneous spacetime of Bianchi type** if it has the following properties:

1. It has a 3-dimensional symmetry group G.
2. The orbits of G are spacelike hypersurfaces in M_4 on which G acts simply transitively.

A Bianchi type spacetime may have a larger symmetry group; the definition then only requires that the full symmetry group has a 3-dimensional subgroup that acts simply transitively on certain spacelike hypersurfaces.

9.5 Invariant vector fields.

As noted in Sec. 8.5, the vector field X is invariant under the group of transformations generated by the vector field k if

$$\underset{k}{\pounds} X \equiv [k, X] = 0. \tag{9.19}$$

Now let X be a vector field on S, let $\underset{(i)}{k}$, $i = 1, \ldots, m$ be a set of generators of invariances of X such that at every point $p \in S$, $\left\{ \underset{(i)}{k}(p) \right\}_{i=1}^{m}$ is a basis in the tangent space to S at p (i. e. the orbits of the group generated by $\underset{(i)}{k}$ are m-dimensional, and so is S). In that case the matrix K where ${K_i}^\alpha = \underset{(i)}{k}{}^\alpha(p)$ is nonsingular at every $p \in S$, and so defines the inverse matrix κ. Let us write (9.19) for all the fields $\underset{(i)}{k}$:

$$\underset{(i)}{k}{}^\rho X^\alpha{}_{,\rho} - X^\rho \underset{(i)}{k}{}^\alpha{}_{,\rho} = 0 \tag{9.20}$$

and multiply this set of equations by the matrix κ. The result will be

$$X^\alpha{}_{,\beta} - \kappa^i{}_\beta \underset{(i)}{k}{}^\alpha{}_{,\rho} X^\rho = 0 \qquad \text{(sum over } i). \tag{9.21}$$

The left-hand side above looks like a covariant derivative of X^α, where

$$G^\alpha{}_{\beta\gamma} \stackrel{\text{def}}{=} -\kappa^i{}_\beta \underset{(i)}{k}{}^\alpha{}_{,\gamma} \qquad \text{(sum over } i\text{)} \tag{9.22}$$

plays the role of the affine connection. Let us follow this analogy and calculate the curvature tensor defined by $G^\alpha{}_{\beta\gamma}$. It turns out to be equal to zero. So, eq. (9.21) defines a transport of the vector field X along the vector fields $\underset{(i)}{k}$ that is formally analogous to the parallel transport on S and has vanishing curvature, i.e., is path-independent. Thus, if $X(p)$ is defined at any point $p \in S$, then (9.21) will uniquely define X at all other points of S.

We can then define a vector *basis* $\underset{(i)}{X}(p)$ at $p \in S, i = 1, \ldots, m$, and use eq. (9.21) (called the **Lie transport**) to transplant the basis to all other point of S. The vector fields $\underset{(i)}{X}$ on S thus obtained will be automatically invariant with respect to the transformations generated by the fields $\underset{(i)}{k}$. We have thus proven the

Theorem 9.1 *If a set of vector fields $\underset{(i)}{k}$ on S exists such that $\underset{(i)}{k}(p)$ form a basis of the tangent space to S at each $p \in S$, then there exists a set of vector fields $\underset{(i)}{X}$ on S that are invariant under the transformations generated by $\underset{(i)}{k}$ and also form a basis at each $p \in S$.*

Now let us assume in addition that all $\underset{(i)}{k}$ are Killing fields, and let us calculate the Lie derivatives along $\underset{(i)}{k}$ of the quantities

$$g_{(j)(k)} \stackrel{\text{def}}{=} \underset{(j)}{X}{}^\alpha \underset{(k)}{X}{}^\beta g_{\alpha\beta}, \tag{9.23}$$

where $g_{\alpha\beta}$ is the metric on S. Since $\underset{(i)}{X}$ are invariant, we have $\underset{\underset{(i)}{k}}{\pounds}\underset{(j)}{X} = 0$, and since $\underset{(i)}{k}$ are now Killing fields, we have $\underset{\underset{(i)}{k}}{\pounds} g_{\alpha\beta} = 0$, so $\underset{\underset{(i)}{k}}{\pounds} g_{(j)(k)} = 0$. But $g_{(j)(k)}$ are scalars, so from (8.52)

$$0 = \underset{\underset{(i)}{k}}{\pounds} g_{(j)(k)} = \underset{(i)}{k}{}^\rho g_{(j)(k),\rho} \tag{9.24}$$

and since $\underset{(i)}{k}$ form a basis, this means that $g_{(j)(k)}$ are constants.

If $\underset{(i)}{X}$ and $\underset{(j)}{X}$ are invariant, then so is $[\underset{(i)}{X}, \underset{(j)}{X}]$. Since $\underset{(i)}{X}(p)$ form a basis at every $p \in S$, it follows that $[\underset{(i)}{X}, \underset{(j)}{X}]$ can be decomposed in this basis,

$$[\underset{(i)}{X}, \underset{(j)}{X}] = D^l{}_{ij} \underset{(l)}{X}, \tag{9.25}$$

where the scalar coefficients $D^l{}_{ij}$ could be expected to depend on the point p. However, calculating the Lie derivative of (9.25) along $\underset{(i)}{k}$, using the invariance of $[\underset{(i)}{X}, \underset{(j)}{X}]$ and of $\underset{(l)}{X}$, and using the fact that $\underset{(l)}{X}$ form a basis at each $p \in S$ we conclude that $D^l{}_{ij}$ are constants.

Let us recall now that the basis $\underset{(i)}{X}$ was defined by choosing $\underset{(i)}{X}(p_0)$ arbitrarily at a certain $p_0 \in S$ and transporting it off p_0 by (9.21). Let us assume then that at p_0 we have $\underset{(i)}{X}^\alpha(p_0) = \underset{(i)}{k}^\alpha(p_0)$. In this case it can be proven that

$$D^l{}_{ij} = -C^l{}_{ij}, \tag{9.26}$$

where $C^l{}_{ij}$ are the structure constants defined by the commutators of $\underset{(i)}{k}$. Hint for the proof: since both $\underset{(i)}{k}$ and $\underset{(i)}{X}$ are bases at every $p \in S$ and $\underset{(i)}{k} = \underset{(i)}{X}$ at p_0, then there exists a matrix $M(\{x\})$ (point-dependent!) such that $\underset{(i)}{X}^\alpha = M_i{}^j \underset{(j)}{k}^\alpha$ (sum over j) and $M_i{}^j(p_0) = \delta_i{}^j$. Knowing this, play with the commutators at p_0. (Without the initial condition $\underset{(i)}{k} = \underset{(i)}{X}$ at p_0, the constants $D^l{}_{ij}$ would be linear combinations of the $C^l{}_{ij}$.)

9.6 The metrics of the Bianchi-type spacetimes.

Let $\underset{(i)}{k}$ be the 3 Killing fields required by the definition, and let x^I be the coordinates in the homogeneous hypersurfaces ($i, I = 1, 2, 3$). These hypersurfaces are tangent to $\underset{(i)}{k}$ (by definition) and uniquely define the vector field m orthogonal to them,

$$m^\alpha = |g|^{-1/2} \epsilon^{\alpha\beta\gamma\delta} \underset{(1)}{k}_\beta \underset{(2)}{k}_\gamma \underset{(3)}{k}_\delta, \tag{9.27}$$

where $g = \det ||g_{\alpha\beta}||$. Let us choose a parameter on the integral lines of m as the t-coordinate: $m^\alpha = \partial x^\alpha / \partial t$. Since $g_{\alpha\beta} m^\alpha \underset{(i)}{k}^\beta = 0$ for all i, in such coordinates we have

$$g_{0I} = 0. \tag{9.28}$$

From the (00) component of the Killing equations, using (9.28) and $\underset{(i)}{k}^0 = 0$, we further obtain $\underset{(i)}{k}^I g_{00,I} = 0$, i. e. $g_{00} = g_{00}(t)$. By the next coordinate transformation, $t' = \int [g_{00}(t)]^{1/2}\, \mathrm{d}t$, we obtain (dropping the prime)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - g_{IJ} \mathrm{d}x^I \mathrm{d}x^J. \tag{9.29}$$

From the $(0, i)$, $i = 1, 2, 3$ components of the Killing equations we then conclude that $\underset{(i)}{k}^I{}_{,0} = 0$ and that $\underset{(i)}{k}$ are Killing fields also for the 3-metric with components g_{IJ}. Since for the homogeneous hypersurfaces with the 3-metric g_{IJ} the fields $\underset{(i)}{k}$ form a basis at each point, we can apply here the results of Section 9.5 to each single hypersurface. Here, however, eq. (9.24) will read $\underset{(i)}{k}^R g_{(j)(k),R} = 0$, but $(\partial/\partial t) g_{(j)(k)}$ is not determined. Consequently, in the Bianchi type spacetimes the $g_{(j)(k)}$ will be functions of t. Let us denote by $\omega^{(i)}{}_I$ the matrix inverse to $\underset{(i)}{X}^I$ (where $\underset{(i)}{X}$ are the vector fields invariant with respect to the transformations generated by $\underset{(i)}{k}$), then, from (9.23):

$$g_{IJ} = g_{(j)(k)}(t) \omega^{(j)}{}_I \omega^{(k)}{}_J \tag{9.30}$$

and

$$ds^2 = dt^2 - g_{(j)(k)}(t)\omega^{(j)}{}_I\omega^{(k)}{}_J \mathrm{d}x^I \mathrm{d}x^J. \tag{9.31}$$

In order to find $\omega^{(i)}{}_I$ one has to construct the Killing fields and their invariant fields for each Bianchi type separately. The resulting formulae can be found e. g. in [37].

9.7 The isotropic Bianchi type spacetimes.

In cosmology, such Bianchi type spacetimes are important that are not only spatially homogeneous, but also spherically symmetric (**isotropic**). We shall deal with their physical and astrophysical implications in chapter 14. Here, we only derive their metric form.

The form (8.50) of the metric is preserved by the transformations $t = f(t', r')$, $r = g(t', r')$ which can be used to simplify (8.50) further. In order to transform (8.50) into the form (9.31) we can choose f and g so that the new $\beta = 0$. In the new form, only those transformations will be permissible which preserve the hypersurfaces $t =$ constant (they are determined geometrically by the symmetry groups existing in the Bianchi models). They are $t = f(t')$, $r = g(r')$. In agreement with (9.29), α should then depend only on t and be transformable to 1. So finally, a spherically symmetric metric can possibly be homogeneous in the Bianchi sense only if it can be put in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \gamma(t,r)\mathrm{d}r^2 + \delta(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2\right). \tag{9.32}$$

With such a metric one should now solve the Killing equations. We will present the solution and give hints on how to derive it later in this chapter.

Some properties of the solution can be guessed in advance. The spacetime with the metric (9.32) has $O(3)$ as a symmetry group. Since $O(3)$ has 2-dimensional orbits, it cannot be the homogeneity group H existing in the Bianchi type spacetimes. Hence, the full symmetry group must contain $O(3)$ and H as two subgroups. $O(3)$ cannot be a subgroup of H because it acts multiply transitively. $O(3)$ and H cannot have any common 2 dimensional subgroup because $O(3)$ has no 2-dimensional subgroups at all. All 1-dimensional subgroups of $O(3)$ are rotations around an axis which act multiply transitively and so cannot be subgroups of H. Thus, $O(3)$ and H cannot have any common subgroup apart from the identity transformation. Hence, the full symmetry group will have *at least* 6 parameters, 3 of them connected with $O(3)$ and 3 with H. On the other hand, we showed in Sec. 8.4 that an n-dimensional manifold can have a symmetry group of *at most* $\frac{1}{2}n(n+1)$ parameters, i. e. at most 6 when $n = 3$. Consequently, the spatially homogeneous and isotropic spacetimes have symmetry groups with exactly 6 parameters.

After the Killing equations for (9.32) are solved and the relevant case is chosen from among the solutions, the metric assumes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R^2(t)}{\left(1 + \frac{1}{4}kr^2\right)^2}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right)\right], \tag{9.33}$$

[37] Chap. 13 and 14 of Ref. [33], in particular Table 13.4 and p. 209.

where $R(t)$ is an arbitrary function and k is an arbitrary constant which, if nonzero, can be scaled to $+1$ or -1 by the transformations of the form $r = Cr'$ (C being constant).

Special cases of this metric, corresponding to $k = +1$ and $k = -1$ were first derived, by a rather loose argument, by A. A. Friedmann [38], who thus, unknowingly, became the father of modern cosmology. (He died before he could witness his success.) A mathematically rigorous derivation, by methods diferent from our Sec. 9.8, and with all signs of k included, was given independently by H. P. Robertson [39, 40] and A. G. Walker [41]. The metric (9.33) is thus frequently named the **Robertson – Walker metric**. (Other names attached to it in various combinations are Friedmann and Lemaître, but those refer to special cases of (9.33); we shall come back to this point in chapter 14.)

9.8 The Robertson–Walker metric – a formal derivation and full groups of symmetries.

We shall now indicate how to solve the Killing equations for the metric (9.33). The calculation is laborious, but uses only routine mathematics. We recall (see Section 9.6) that in a general Bianchi-type spacetime the Killing fields have no time-component ($k^0 = 0$), while the components (00) and $(0, i)$, $i = 1, 2, 3$, of the Killing equations have already been solved with the result $k^I{}_{,t} = 0$, $I = 1, 2, 3$. (We could proceed without using this information, but then subcases with 4-dimensional orbits of the symmetry group – the Minkowski and de Sitter metrics – would separately appear.)

The remaining Killing equations are

$$\begin{aligned}
k^1\gamma_{,r} + 2k^1{}_{,r}\,\gamma &= 0, &(9.34)\\
k^2{}_{,r}\,\delta + k^1{}_{,\vartheta}\,\gamma &= 0, &(9.35)\\
k^3{}_{,r}\,\delta\sin^2\vartheta + k^1{}_{,\varphi}\,\gamma &= 0, &(9.36)\\
k^1\delta_{,r} + 2k^2{}_{,\vartheta}\,\delta &= 0, &(9.37)\\
k^3{}_{,\vartheta}\sin^2\vartheta + k^2{}_{,\varphi} &= 0, &(9.38)\\
k^1\delta_{,r} + 2k^2\cot\vartheta\delta + 2k^3{}_{,\varphi}\,\delta &= 0. &(9.39)
\end{aligned}$$

We seek solutions with the physical signature $(+ - --)$, so $\gamma < 0$ and $\delta < 0$. We also demand that $k^1 \neq 0$, for otherwise the orbits of the symmetry group would come out 2-dimensional, contrary to a basic property of the Bianchi-type spacetimes. The set (9.34) – (9.39) is overdetermined, so, whichever equation we solve first, the solution will be further limited by the remaining equations. Limitations are thereby imposed not only on the Killing fields, but also on the metric components γ and δ. Along the way, some of the

[38] A. A. Friedmann, *Z. Physik* **10**, 377 (1922); **21**, 326 (1924); English translation: *Gen. Relativ. Gravit.* **31**, 1991 (1999); with an editorial note by G. F. R. Ellis and A. Krasiński, *Gen. Relativ. Gravit.* **31**, 1985 (1999), and author's biography by A. Krasiński, *Gen. Relativ. Gravit.* **31**, 1989 (1999). See also an addendum: *Gen. Relativ. Gravit.* **32**, 1937 (2000).

[39] H. P. Robertson, *Proc. Nat. Acad. Sci. USA* **15**, 822 (1929).

[40] H. P. Robertson, *Rev. Mod. Phys.* **5**, 62 (1933).

[41] A. G. Walker, *Quart. J. Math. Oxford*, ser. 6, 81 (1935).

equations lead to alternatives of the type ($ab = 0 \Longrightarrow a = 0$ or $b = 0$), and this is where the special cases appear.

One of the special solutions that emerge is the metric[5]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\mathrm{d}r^2 - S^2(t)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right), \tag{9.40}$$

whose generators of symmetries are

$$\underset{(1)}{J} = \frac{\partial}{\partial r}, \qquad \underset{(2)}{J} = \cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi},$$
$$\underset{(3)}{J} = \sin\varphi\frac{\partial}{\partial\vartheta} + \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}, \qquad \underset{(4)}{J} = \frac{\partial}{\partial\varphi}. \tag{9.41}$$

The operators $\underset{(2)}{J}$, $\underset{(3)}{J}$ and $\underset{(4)}{J}$ generate the assumed $O(3)$ group. The 4-parameter group generated by all of (9.41) has 3-dimensional orbits, and has no 3-parameter simply transitive subgroup. This is because $O(3)$ alone has 2-dimensional orbits and so cannot be simply transitive, and has no 2-dimensional subgroups that could be combined with the transformations generated by $\underset{(1)}{J}$ into a 3-parameter group. Hence, (9.40) does not belong to the Bianchi class. It is a metric of the Kantowski–Sachs class [15].[6]

The generic solution of (9.34) – (9.39) is

$$\begin{aligned}
\underset{(1)}{J} &= -V\cos\vartheta\frac{\partial}{\partial r} + W\sin\vartheta\frac{\partial}{\partial\vartheta},\\
\underset{(2)}{J} &= V\sin\vartheta\cos\varphi\frac{\partial}{\partial r} + W\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - W\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi},\\
\underset{(3)}{J} &= V\sin\vartheta\sin\varphi\frac{\partial}{\partial r} + W\cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} + W\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi},
\end{aligned}$$

[5] A transformation of the r coordinate is required to achieve the form (9.40).

[6] Metrics with the Kantowski-Sachs symmetry have a longer history than most people suspect. A *generalisation* of such a metric first appeared in a paper by Datt

[42] B. Datt, *Z. Physik* **108**, 314 (1938); English translation: *Gen. Relativ. Gravit.* **31**, 1619 (1999), with an editorial note by A. Krasiński, *Gen. Relativ. Gravit.* **31**, 1615 (1999),

but the author instantly dismissed it as unphysical. Some physical properties of the metrics (9.40) were investigated by Kompaneets and Chernov

[43] A. S. Kompaneets and A. S. Chernov, *ZhETF* **47**, 1939 (1964); English translation: *Sov. Phys. JETP* **20**, 1303 (1965).

The symmetry was noted and investigated by Kantowski,

[44] R. Kantowski, *PhD Thesis*, reprinted in *Gen. Relativ. Gravit.* **30**, 1665 (1998), with an editorial note by A. Krasiński, *Gen. Relativ. Gravit.* **30**, 1663 (1998) and author's biography by R. Kantowski, *Gen. Relativ. Gravit.* **30**, 1663 (1998).

and became a classical piece of knowledge after the paper by Kantowski and Sachs [15]. The geometric properties of the Datt metric were first investigated by Ruban

[45] V. A. Ruban, *Pisma v Red. ZhETF* **8**, 669 (1968); English translation: *Sov. Phys. JETP Lett.* **8**, 414 (1968); reprinted in *Gen. Relativ. Gravit.* **33**, 369 (2001),

[46] V. A. Ruban, *ZhETF* **56**, 1914 (1969); English translation: *Sov. Phys. JETP* **29**, 1027 (1969); reprinted in *Gen. Relativ. Gravit.* **33**, 375 (2001), with an editorial note by A. Krasiński, *Gen. Relativ. Gravit.* **33**, 363 (2001), and author's biography by I. Dymnikova, *Gen. Relativ. Gravit.* **33**, 366 (2001).

See the editorial notes to the Datt, Kantowski and Ruban papers for more on this story.

$$\begin{aligned}
\underset{(4)}{J} &= \cos\varphi\frac{\partial}{\partial\vartheta} - \cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi}, \\
\underset{(5)}{J} &= \sin\varphi\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi}, \qquad \underset{(6)}{J} = -\frac{\partial}{\partial\varphi},
\end{aligned} \tag{9.42}$$

where

$$W(r) \stackrel{\text{def}}{=} \frac{1}{r} - \frac{1}{4}kr, \qquad V(r) \stackrel{\text{def}}{=} 1 + \frac{1}{4}kr^2. \tag{9.43}$$

The metric with these symmetries is the one given by (9.33).

The last three generators in (9.42) are easily recognised as those of $O(3)$. We shall use the abbreviation $[i,j] \stackrel{\text{def}}{=} [\underset{(i)}{J}, \underset{(j)}{J}]$. The commutators are

$$\begin{array}{lll}
[1,2] = k\underset{(4)}{J} & & \\
[1,3] = k\underset{(5)}{J} & [2,3] = k\underset{(6)}{J} & \\
[1,4] = -\underset{(2)}{J} & [2,4] = \underset{(1)}{J} & [3,4] = 0 \\
[1,5] = -\underset{(3)}{J} & [2,5] = 0 & [3,5] = \underset{(1)}{J} \\
[1,6] = 0 & [2,6] = -\underset{(3)}{J} & [3,6] = \underset{(2)}{J} \\
[4,5] = \underset{(6)}{J} & [4,6] = -\underset{(5)}{J} & [5,6] = \underset{(4)}{J}.
\end{array} \tag{9.44}$$

The last three commutators are those of the algebra of $O(3)$. In (9.44) such relations should be found that correspond to the Bianchi algebras. However, the Bianchi classification introduced standard bases, and (9.42) may contain transformed generators. Indeed, some of the Bianchi bases are linear combinations of (9.42).

In the standard Bianchi bases, the nonzero structure constants were scaled to $+1$ or -1 whenever possible. Thus, for comparing (9.44) with (9.15) – (9.16) and Table 9.2, we have to scale out k in (9.44); the scaling is $\left(\underset{(1)}{J}, \underset{(2)}{J}, \underset{(3)}{J}\right) = \sqrt{|k|}\left(\underset{(1)}{\widetilde{J}}, \underset{(2)}{\widetilde{J}}, \underset{(3)}{\widetilde{J}}\right)$, with other generators unchanged. The result is as if $k = -1$ when $k < 0$ and $k = +1$ when $k > 0$. The formulae below show the scaled generators, with tildes omitted.

For $k > 0$, the Bianchi sub-basis is of type IX:

$$\underset{(1)}{L} = \frac{1}{2}\left(\underset{(1)}{J} + \underset{(6)}{J}\right), \qquad \underset{(2)}{L} = \frac{1}{2}\left(\underset{(2)}{J} - \underset{(5)}{J}\right), \qquad \underset{(3)}{L} = \frac{1}{2}\left(\underset{(3)}{J} + \underset{(4)}{J}\right). \tag{9.45}$$

The new generators are determined up to arbitrary orthogonal transformations because the matrix n^{ij} has a triple eigenvalue.

For $k < 0$, two Bianchi algebras are found in (9.42), one of type V:

$$\underset{(1)}{L} = -\underset{(1)}{J}, \qquad \underset{(2)}{L} = \underset{(2)}{J} + \underset{(4)}{J}, \qquad \underset{(3)}{L} = \underset{(3)}{J} + \underset{(5)}{J}, \tag{9.46}$$

and one of type VII_h:

$$\underset{(1)}{l} = -a\underset{(1)}{J} + \underset{(6)}{J}, \qquad \underset{(2)}{l} = \underset{(2)}{J} + \underset{(4)}{J}, \qquad \underset{(3)}{l} = \underset{(3)}{J} + \underset{(5)}{J}. \tag{9.47}$$

For $k = 0$, two standard Bianchi bases are contained in (9.42): $\left\{ \underset{(1)}{J}, \underset{(2)}{J}, \underset{(3)}{J} \right\}$ of Bianchi type I and $\left\{ \underset{(1)}{J}, \underset{(2)}{J}, \underset{(4)}{J} \right\}$ of Bianchi type VII_0.

It is easy to verify that the examples of bases shown are indeed the standard Bianchi bases of the types indicated. It is, however, more difficult to prove that only these Bianchi algebras are contained in (9.42). This fact was apparently first discovered by Grishchuk [47]. The relation between the symmetries of (9.34) and the possible Bianchi groups is discussed in more detail in [20].

9.9 Exercises

1. Verify that the curvature defined by the "connection" (9.22) is indeed zero.

2. Verify eq. (9.26).

3. Since the invariant fields are defined by $[\underset{(i)}{k}, X] = 0$, and the Killing fields $\underset{(i)}{k}$ do not depend on t, we can solve this set of equations assuming that the invariant fields are independent of time, too. However, a general solution of the equations $[\underset{(i)}{k}, X] = 0$ for X does depend on time. Show that the time-dependent fields are of the form

$$\underset{(j)}{X}{}^I(t) = B_j{}^k(t) \underset{(k)}{X}{}^I(t_0), \tag{9.48}$$

where t_0 is some initial value and $B_j{}^k(t)$ is a nonsingular 3×3 matrix. Hence, even if the invariant fields are chosen as time-dependent, the whole time dependence can be hidden in the coefficients $\widetilde{g}_{(j)(k)}(t) = g_{(r)(s)}(t) {B^{-1}}^r{}_j {B^{-1}}^s{}_k$, and we can use the time-independent $X^I(t_0)$ as the new invariant fields.

Hint: Show first that if X is an invariant field, then so is $\mathrm{d}X/\mathrm{d}t$, and so $\mathrm{d}X/\mathrm{d}t$ can be decomposed in the basis of the invariant fields (with time-dependent coefficients).

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[47] L. P. Grishchuk, *Astron. Zh.* **44**, 1097 (1967); English translation: *Sov. Astron. A. J.* **11**, 881 (1968).