Chapter 12

Relativistic hydrodynamics

12.1 Motion of a continuous medium in Newtonian mechanics

Let x_i, $i = 1, 2, 3$ be rectangular Cartesian coordinates in a Euclidean 3-space. Let one flow line of a fluid pass through every point of a certain region in space. Let the velocity field of the fluid, $v_i(t, x_j)$, be differentiable at every point x_i and at every instant t. Then

$$\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} = v_j(t, y_i)\big|_{y_j = x_j}. \tag{12.1}$$

The $x_j(t)$ on the left are coordinates of the flowing fluid particle, while the y_i on the right are the coordinates of a point of space.

Consider a particle P which at the instant t occupies the position $\{x_i\}$ and moves with the velocity $v_j(t, x_i)$, and an adjacent particle Q that, *at the same instant* t, occupies the position $\{x_i + \delta x_i\}$ and moves with the velocity $v_j(t, x_i + \delta x_i)$. Up to terms linear in δx_i, the velocity of Q relative to P is

$$(v_{QP})_j(t) \equiv v_j(t, x_i + \delta x_i) - v_j(t, x_i) = v_{j,k}(t, x_i)\delta x_k + O\left((\delta x)^2\right) \tag{12.2}$$

(sums over repeated Latin indices are implied throughout this section), where $O\left((\delta x)^2\right)$ are terms of order ≥ 2 in δx_k. Hence, at the instant $(t + \Delta t)$ the position of Q relative to P will be given by the vector

$$\delta x'_j = \delta x_j + (v_{QP})_j\,\Delta t + O\left((\Delta t)^2\right) = \delta x_j + v_{j,k}\delta x_k \Delta t + O\left((\Delta t)^2, (\delta x)^2\right). \tag{12.3}$$

Thus, the matrix $v_{j,k}$ determines the relative velocity of two neighbouring particles of the fluid. Under transformations between Cartesian coordinates, $v_{j,k}$ transforms as a tensor. It can be decomposed into three parts, each of which transforms independently of the others:

$$v_{j,k} = \sigma_{jk} + \omega_{jk} + \frac{1}{3}\delta_{jk}\theta, \qquad \text{where} \tag{12.4}$$

$$\theta \stackrel{\text{def}}{=} v_{j,j}, \tag{12.5}$$

$$\sigma_{jk} \stackrel{\text{def}}{=} v_{(j,k)} - \frac{1}{3}\delta_{jk}\theta, \tag{12.6}$$

$$\omega_{jk} \stackrel{\text{def}}{=} v_{[j,k]}. \tag{12.7}$$

This decomposition can be done for every tensor of rank 2. For $v_{j,k}$, each part has a physical interpretation. To read it out, let us consider the following three types of motion:

- I. Let $\sigma_{jk} = \omega_{jk} = 0 \neq \theta$. Then, from (12.2) and (12.3):

$$\delta x'_j = \left(1 + \frac{1}{3}\theta\Delta t\right)\delta x_j + O\left((\Delta t)^2, (\delta x)^2\right). \tag{12.8}$$

The vector that connects P and Q at the instant $(t + \Delta t)$ has the same direction as the one that connected them at t, but a different length, and $\theta = 3\,(\mathrm{d}/\mathrm{d}t)(\ln|\delta\mathbf{x}|)$. In this type of motion, P and Q either recede from each other (when $\theta > 0$) or approach each other (when $\theta < 0$) along the straight line PQ. Such motion is called **isotropic expansion** or **contraction**, and θ is called the **scalar of expansion**.

- II. Let $\theta = 0$, and $\sigma_{jk} = 0 \neq \omega_{jk}$. Then, from (12.3) and (12.4):

$$\delta x'_i = (\delta_{ik} + \omega_{ik}\Delta t)\,\delta x_k + O\left((\Delta t)^2, (\delta x)^2\right). \tag{12.9}$$

For the length of the vector $\delta x'_i$, in consequence of antisymmetry of ω_{ik}, we get

$$\begin{aligned}\delta\ell' &= (\delta x'_i\delta x'_i)^{1/2} = \left[\delta x_k\delta x_k + O\left((\Delta t)^2, (\delta x)^3\right)\right]^{1/2}\\ &= (\delta x_k\delta x_k)^{1/2} + O\left((\Delta t)^2, (\delta x)^3\right) = \delta\ell + O\left((\Delta t)^2, (\delta x)^3\right).\end{aligned} \tag{12.10}$$

Thus, up to terms of order $(\Delta t)^2$ and $(\delta x)^3$, the length of the vector δx_i does not change. Let us see what happens with the direction of δx_i:

$$(\delta x'_i - \delta x_i)\,\delta x_i = \omega_{ik}\delta x_i\delta x_k\Delta t + O\left((\Delta t)^2, (\delta x)^3\right) = O\left((\Delta t)^2, (\delta x)^3\right). \tag{12.11}$$

So, at the same level of approximation, the change in δx_i is perpendicular to δx_i. The properties (12.10) and (12.11) are characteristic for rotational motion. Hence, in this type of motion, Q revolves around P, with the angular velocity $\vec{\omega}$ obeying

$$\mathbf{v}_{QP} = \vec{\omega} \times \delta\mathbf{x} \Longrightarrow (v_{QP})_i = \epsilon_{ikl}\omega_k\delta x_l. \tag{12.12}$$

But from (12.2) and (12.4) $(v_{QP})_i = \omega_{il}\delta x_l$, so

$$\omega_{il} = -\epsilon_{ilk}\omega_k. \tag{12.13}$$

Inverting this equation and using (12.7) we get

$$\omega_j = \frac{1}{2}\epsilon_{jil}\omega_{li} = \frac{1}{2}\epsilon_{jil}v_{l,i} \Longrightarrow \vec{\omega} = \frac{1}{2}\mathrm{rot}\ \mathbf{v}. \tag{12.14}$$

In consequence of (12.13) and (12.14), ω_{ik} is called the **rotation tensor**. Note that $\omega^2 = \omega_i\omega_i = \frac{1}{2}\omega_{kl}\omega_{kl} \geq 0$ and vanishes only when $\omega_{kl} = 0$. Hence, also the **scalar of rotation** ω can be used to differentiate between rotational and irrotational motion.

- III. Let $\theta = 0$ and $\omega_{ij} = 0 \neq \sigma_{ij}$. Consider three particles Q_1, Q_2 and Q_3 occupying, at the instant t, the positions relative to P given by the vectors $\delta\mathbf{x}$, $\delta\mathbf{y}$ and $\delta\mathbf{z}$ attached to P. The volume of the parallelepiped spanned on $\delta\mathbf{x}$, $\delta\mathbf{y}$ and $\delta\mathbf{z}$ at t is

$$\delta V = \delta\mathbf{x} \cdot (\delta\mathbf{y} \times \delta\mathbf{z}) = \epsilon_{ijk}\delta x_i \delta y_j \delta z_k. \tag{12.15}$$

At the instant $(t + \Delta t)$, the corresponding volume will be equal to

$$\delta V' = \delta V + D\Delta t + O\left((\Delta t)^2, (\delta x)^4\right), \tag{12.16}$$

where

$$D \stackrel{\text{def}}{=} \left(\epsilon_{lmk}\sigma_{kn} + \epsilon_{ljn}\sigma_{jm} + \epsilon_{imn}\sigma_{il}\right)\delta x_l \delta y_m \delta z_n. \tag{12.17}$$

Using $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{ii} = 0$ one can now verify that $D = 0$, i.e. $\delta V' = \delta V + O\left((\Delta t)^2, (\delta x)^4\right)$. Hence, in this type of motion $V =$ constant up to terms of order $O\left((\Delta t)^2, (\delta x)^4\right)$. However, the shape of the parallelepiped changes because the vector $\sigma_{ik}\delta x_k$ has in general a different direction[28] and different length from δx_i. This is called **shearing motion**, and σ_{ij} is called the **shear tensor**. Similarly to rotation, the $\sigma_{ij} = 0$ if and only $\sigma^2 \stackrel{\text{def}}{=} \frac{1}{2}\sigma_{ij}\sigma_{ij} = 0$; the scalar σ is called simply **shear**.

12.2 Motion of a continuous medium in relativistic mechanics.

In relativity, we also assume that one flow line of a fluid passes through every point $\{x^\alpha\}$ of a certain region in spacetime, and that the velocity field of the fluid, $u^\alpha(x)$ is differentiable in this whole region. Then $\mathrm{d}x^\alpha/\mathrm{d}s = u^\alpha(x^\beta)$, and, just as in (12.1), the $x^\alpha(s)$ on the left-hand side are coordinates of the flowing fluid element, while the x^β on the right-hand side are the coordinates of that point of spacetime in which $x^\alpha(s)$ equals x^α. The parameter s is the proper time on the world-lines of the fluid, hence

$$u_\alpha(x)u^\alpha(x) = 1. \tag{12.18}$$

Note now that the tensor

$$h^\alpha{}_\beta \stackrel{\text{def}}{=} \delta^\alpha{}_\beta - u^\alpha u_\beta \tag{12.19}$$

projects vectors onto a hypersurface orthogonal to u^α at a given point x.[29] This is because $h^\alpha{}_\beta u^\beta = 0$, and for an arbitrary vector B^α, $u_\alpha\left(h^\alpha{}_\beta B^\beta\right) = 0$. The quantity $B^\alpha_\perp \stackrel{\text{def}}{=} h^\alpha{}_\beta B^\beta$ is the component of B^α along the direction perpendicular to u_α. Note also that

$$g_{\alpha\beta}B^\alpha_\perp B^\beta_\perp = h_{\alpha\beta}B^\alpha_\perp B^\beta_\perp. \tag{12.20}$$

[28] If $\delta\mathbf{x}$, $\delta\mathbf{y}$ and $\delta\mathbf{z}$ are collinear with the eigenvectors of the matrix σ, then their directions do not change during the motion, but, in consequence of $\sigma_{ii} \equiv \mathrm{Tr}(\sigma) = 0$, the sum of changes of their lengths must be zero. Consequently, the shape of the parallelepiped will be changed also in this case.

[29] Each of the hypersurfaces meant here is orthogonal to a *single* flow-line of the fluid. This family of hypersurfaces is in general different for every flow-line. A vector field is orthogonal to a family of hypersurfaces if and only if its *rotation*, defined further in this section, is zero – see Sec. 15.1.

Thus, $h_{\alpha\beta}$ plays the role of the metric tensor in the hypersurfaces orthogonal to u^α.

If the x^0-coordinate in spacetime is chosen as the proper time s on the flow lines, then

$$u'^\alpha \underset{*}{=} \delta^\alpha{}_0. \tag{12.21}$$

Let us choose the spatial coordinates x^I, $I = 1, 2, 3$ so that their parametric lines are contained in the hypersurfaces $S_P(s)$ orthogonal to one fixed worldline P. In such coordinates $g_{0I}|_P \underset{*}{=} 0$, so for a vector field B^α orthogonal to $u^\alpha_P(s)$ we have $0 = \left(g_{\alpha\beta}u^\alpha B^\beta\right)_P \underset{*}{=} g_{00}|_P B^0_P$, i.e. $B^0_P \underset{*}{=} 0$ (since $g_{00} \neq 0$ from (12.21) and (12.18)). Consider a curve tangent to B^α, and let λ be a parameter on it. In our chosen coordinats we have $0 \underset{*}{=} B^0_P = \mathrm{d}x^0/\mathrm{d}\lambda|_P$, i.e. at P these lines are tangent to the hypersurface $x^0 =$ constant. Consequently, $S_P(s_0)$ is called the **hypersurface of events simultaneous with** $P(s_0)$ or the **hypersurface of constant time** $s = s_0$ for the observer P.

Let a particle moving along the curve P occupy at the instant s the point P_0 in spacetime. Let Q be an adjacent worldline, and let δx^α be a vector joining P_0 to an arbitrary point on Q. The event Q_0 simultaneous with P_0 is then at the position relative to P_0 given by the vector $\delta_\perp x^\alpha = h^\alpha{}_\beta\,(P_0)\,\delta x^\beta$. The velocity of the fluid at P_0 is $u^\alpha(x^\beta)$, and the velocity at Q_0 is $u^\alpha(x^\beta + \delta_\perp x^\beta)$. After the time Δs, the particle that at s was at P_0 will be at P_1 of coordinates $x'^\alpha = x^\alpha + u^\alpha \Delta s$. Where will then be the particle that occupied the point Q_0 at s? Its position relative to P_0 will be determined by a vector of the form $\delta_\perp x^\alpha + v^\alpha \Delta s$, where v^α is determined by $u^\alpha(x^\beta + \delta_\perp x^\beta)$. However, v^α cannot be equal to $u^\alpha(x^\beta + \delta_\perp x^\beta)$, because the latter is attached to the point of coordinates $x^\beta + \delta_\perp x^\beta$, while $\delta_\perp x^\alpha$ is attached to P_0 of coordinates x^β. In order to add the vectors, we must transport one of them parallely to the point of attachment of the other. Thus v^α must be the vector $u^\alpha(x^\beta + \delta_\perp x^\beta)$ parallely transported from $(x^\beta + \delta_\perp x^\beta)$ back to x^β.

Let us apply eq. (5.6) of parallel transport to our present case. The $v^\alpha_\parallel(\tau_2)$ of (5.6) is our v^α, and the $v^\alpha(\tau_1)$ of (5.6) is our $u^\alpha(x^\beta + \delta_\perp x^\beta)$. Hence we have

$$\begin{aligned} v^\alpha &= u^\alpha(x^\beta + \delta_\perp x^\beta) - \int_{x^\beta + \delta_\perp x^\beta}^{x^\beta} \Gamma^\alpha{}_{\sigma\rho}(x) u^\sigma(x) \mathrm{d}x^\rho \\ &\equiv u^\alpha(x^\beta + \delta_\perp x^\beta) + \int_{x^\beta}^{x^\beta + \delta_\perp x^\beta} \Gamma^\alpha{}_{\sigma\rho}(x) u^\sigma(x) \mathrm{d}x^\rho \end{aligned}$$

We now apply the mean value theorem to the integral, and also expand $u^\alpha(x^\beta + \delta_\perp x^\beta)$ by the Taylor formula up to terms linear in $\delta_\perp x^\beta$. The result is

$$v^\alpha = u^\alpha(x^\beta) + u^\alpha{}_{,\rho}\,(x^\beta)\delta_\perp x^\rho + \Gamma^\alpha{}_{\sigma\rho}(\overline{x}) u^\sigma(\overline{x}) \delta_\perp x^\rho + O\left((\delta_\perp x)^2\right),$$

where $x^\beta \leq \overline{x}^\beta \leq (x^\beta + \delta_\perp x^\beta)$. When we replace $\overline{x}^\beta$ by x^β, the difference will be of the order of $\delta_\perp x^\beta$, and since the whole expression is multiplied by $\delta_\perp x^\beta$, the difference in the equation will be of order $O\left((\delta_\perp x)^2\right)$ – which we neglect anyway. Thus

$$\begin{aligned} v^\alpha &= u^\alpha(x^\beta) + u^\alpha{}_{,\rho}\,(x^\beta)\delta_\perp x^\rho + \Gamma^\alpha{}_{\sigma\rho}(x) u^\sigma(x) \delta_\perp x^\rho + O\left((\delta_\perp x)^2\right) \\ &= u^\alpha(x^\beta) + u^\alpha{}_{;\rho}\,(x^\beta)\delta_\perp x^\rho + O\left((\delta_\perp x)^2\right). \end{aligned} \tag{12.22}$$

Hence, the particle that occupied the point Q_0 at the instant s will, at $(s+\Delta s)$, occupy the point of coordinates

$$x''^{\alpha} = x^{\alpha} + \delta_{\perp}x^{\alpha} + u^{\alpha}(x^{\beta})\Delta s + u^{\alpha};_{\rho}(x^{\beta})\delta_{\perp}x^{\rho}\Delta s + O\left((\delta_{\perp}x)^2\right). \tag{12.23}$$

The new position of Q relative to P will thus be given by the vector

$$\delta_{\perp}x'^{\alpha} = x''^{\alpha} - x'^{\alpha} = \delta_{\perp}x^{\alpha} + u^{\alpha};_{\rho}\,\delta_{\perp}x^{\rho}\Delta s + O\left((\delta_{\perp}x)^2\right). \tag{12.24}$$

Thus, the matrix $u^{\alpha};_{\rho}$ determines the rate of change of position of the particle Q with respect to P. However, not the whole matrix $u^{\alpha};_{\rho}$ contributes to (12.24). We have

$$u^{\alpha};_{\rho} \equiv u^{\alpha};_{\sigma}\,\delta^{\sigma}{}_{\rho} \equiv u^{\alpha};_{\sigma}\left(h^{\sigma}{}_{\rho} + u^{\sigma}u_{\rho}\right). \tag{12.25}$$

Only the first term in (12.25) gives a nonzero contribution in (12.24). Hence, finally

$$\delta_{\perp}x'^{\alpha} = \delta_{\perp}x^{\alpha} + u^{\alpha};_{\sigma}\,h^{\sigma}{}_{\rho}\delta_{\perp}x^{\rho}\Delta s + O\left((\delta_{\perp}x)^2\right). \tag{12.26}$$

We now rewrite the final form of (12.25) and (10.67):

$$u^{\alpha};_{\sigma} \equiv h^{\alpha}{}_{\mu}u^{\mu};_{\sigma}\,, \qquad u^{\alpha};_{\sigma}\,u_{\alpha} \equiv 0. \tag{12.27}$$

They show that $u^{\alpha};_{\sigma}\,h^{\sigma}{}_{\rho}$ is an operator acting in a 3-dimensional hypersurface orthogonal to u^{α}. Comparing (12.26) with (12.3) we see that it plays in relativistic hydrodynamics the same role that $v_{j,k}$ played in Newtonian hydrodynamics. This time, $u^{\alpha};_{\sigma}\,h^{\sigma}{}_{\beta}$ is a genuine tensor, and we can decompose it into independent parts in the same way:

$$\begin{aligned} u_{\alpha;\sigma}h^{\sigma}{}_{\beta} &\equiv u_{\alpha;\beta} - \dot{u}_{\alpha}u_{\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}, \qquad \text{where} && (12.28)\\ \dot{u}_{\alpha} &\stackrel{\text{def}}{=} u_{\alpha;\beta}u^{\beta}, && (12.29)\\ \theta &= u^{\alpha};_{\sigma}\,h^{\sigma}{}_{\alpha} \equiv u^{\alpha};_{\alpha}\,, && (12.30)\\ \omega_{\alpha\beta} &= u_{[\alpha;|\sigma|}h^{\sigma}{}_{\beta]} \equiv u_{[\alpha;\beta]} - \dot{u}_{[\alpha}u_{\beta]}, && (12.31)\\ \sigma_{\alpha\beta} &= u_{(\alpha;|\sigma|}h^{\sigma}{}_{\beta)} - \frac{1}{3}\theta h_{\alpha\beta} \equiv u_{(\alpha;\beta)} - \dot{u}_{(\alpha}u_{\beta)} - \frac{1}{3}\theta h_{\alpha\beta}. && (12.32)\end{aligned}$$

These quantities are called, respectively, the **acceleration vector**, the **scalar of expansion**, the **rotation tensor** and the **shear tensor** [101]. The vector $\dot{u}^{\alpha}$ is called acceleration because u^{α} is geodesic when $\dot{u}^{\alpha} = 0$, i.e. the fluid moves then under the influence of gravitation only, and this means free motion in the language of relativity. Equations (12.27), (12.30) and (12.32) imply

$$\sigma_{\alpha\beta}u^{\beta} = \omega_{\alpha\beta}u^{\beta} = 0. \tag{12.33}$$

The tensor of rotation can be uniquely represented by the vector field

$$w^{\alpha} = \frac{1}{2\sqrt{-g}}\,\epsilon^{\alpha\beta\gamma\delta}u_{\beta}\omega_{\gamma\delta}, \tag{12.34}$$

[101] J. Ehlers, *Abhandlungen der Mathematisch-Naturwissenschaftlichen Klasse der Akademie der Wissenschaften und Literatur Mainz*, No 11 (1961); English translation: *Gen. Relativ. Gravit.* **25**, 1225 (1993).

and the rotational motion may be characterised by the scalar ω:

$$\omega^2 = -w_\alpha w^\alpha = \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \geq 0, \tag{12.35}$$

called the **rotation scalar**. Similarly, the shearing motion may be characterised by the **shear scalar** σ defined by

$$\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0. \tag{12.36}$$

The quantities (12.30), (12.31) and (12.32) are, in suitably chosen coordinates, proportional to their Newtonian counterparts given by (12.5) – (12.7). Namely, at a fixed point p_0 of the spacetime let us choose such coordinates in which $\left\{ {\alpha \atop \beta\gamma} \right\}(p_0) = 0$, i.e. $\left.\mathrm{D}/\partial x^\alpha\right|_{p_0} = \partial/\partial x^\alpha$ (we showed in Sec. 10.2 that such coordinates exist along every non-null geodesic, so they certainly exist at a single point). Then, after carrying out all calculations in (12.30) – (12.32), one must substitute $v/c = 0$. Marking the relativistic quantities with the subscript R and their Newtonian counterparts with the subscript N, we get

$$\begin{array}{llll} c\theta_R = \theta_N, & c\underset{R}{\omega}_{ij} = \underset{N}{\omega}_{ij}, & c\underset{R}{\sigma}_{ij} = \underset{N}{\sigma}_{ij}; \\ & \underset{R}{\omega}_{0i} \underset{v/c\to 0}{\longrightarrow} 0, & \underset{R}{\sigma}_{0i} \underset{v/c\to 0}{\longrightarrow} 0. \end{array} \tag{12.37}$$

12.3 The equations of evolution of θ, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$ and $\dot{u}^\alpha$; the Raychaudhuri equation.

The equations derived in this section are consistency conditions between the curvature of spacetime and the hydrodynamical quantities defined above.[30]

Equation (6.5) applied to the velocity field of a fluid is

$$u_{\gamma;\delta\sigma} - u_{\gamma;\sigma\delta} = -R_{\gamma\rho\delta\sigma}u^\rho. \tag{12.38}$$

Let us contract both sides of this with $u^\sigma {h^\gamma}_\alpha {h^\delta}_\beta$. In the second term on the left-hand side we then transfer the derivative with the index δ from $u_{\gamma;\sigma}$ to u^σ, while on the right-hand side we use the antisymmetries of the Riemann tensor. The result is

$${h^\gamma}_\alpha {h^\delta}_\beta \left(u_{\gamma;\delta}\right)^{\cdot} - {h^\gamma}_\alpha {h^\delta}_\beta \dot{u}_{\gamma;\delta} + {h^\gamma}_\alpha {h^\delta}_\beta {u^\sigma}_{;\delta}\, u_{\gamma;\sigma} = -R_{\alpha\rho\beta\sigma}u^\rho u^\sigma. \tag{12.39}$$

[30] In the original paper in which these equations were first derived

[102] G. F. R. Ellis, in: *Proceedings of the International School of Physics "Enrico Fermi", Course 47: General Relativity and Cosmology.* Edited by R. K. Sachs. Academic Press, New York and London, p. 104 (1971); reprinted in: *Gen. Relativ. Gravit.* **41**, 581 (2009), with an editorial note by W. Stoeger, *Gen. Relativ. Gravit.* **41**, 575 (2009) and author's (auto)biography by G. F. R. Ellis, *Gen. Relativ. Gravit.* **41**, 578 (2009),

and in probably all papers in which they were applied (e.g.

[103] A. Barnes, *Gen. Relativ. Gravit.* **4**, 105 (1973)),

the signature $(-+++)$ was used, as opposed to $(+---)$ used in this text. This is why the equations of this section will in some details differ from those of Refs. [102, 103].

where $(u_{\gamma;\delta})^{\cdot} \stackrel{\text{def}}{=} u^\mu \nabla_\mu (u_{\gamma;\delta})$. We contract (12.39) with $g^{\alpha\beta}$ and obtain

$$h^{\gamma\delta}(u_{\gamma;\delta})^{\cdot} - h^{\gamma\delta}\dot{u}_{\gamma;\delta} + h^{\gamma\delta}u^\sigma{}_{;\delta}\, u_{\gamma;\sigma} + R_{\rho\sigma}u^\rho u^\sigma = 0. \tag{12.40}$$

Now we use the Einstein equations with a perfect fluid source to replace

$$R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) = \kappa\left[(\epsilon+p)u_\alpha u_\beta + \frac{1}{2}(p-\epsilon)g_{\alpha\beta}\right], \tag{12.41}$$

and then we apply the decomposition (12.28) to the term $h^{\gamma\delta}u^\sigma{}_{;\delta}$. In the two remaining terms of (12.40) we substitute the definition of $h^{\gamma\delta}$ from (12.19) and we transfer the differentiation from the derivatives of u_γ to u^γ. The result is

$$\begin{aligned}(u^\gamma{}_{;\gamma})^{\cdot} &- (u^\gamma u_{\gamma;\delta})^{\cdot}\, u^\delta + \dot{u}^\gamma u^\delta u_{\gamma;\delta} - \dot{u}^\gamma{}_{;\gamma} + (u^\gamma \dot{u}_\gamma)_{;\delta}\, u^\delta \\ &- u^\gamma{}_{;\delta}\, u^\delta \dot{u}_\gamma + u_{\gamma;\sigma}\left(\sigma^{\sigma\gamma} + \omega^{\sigma\gamma} + \frac{1}{3}\theta h^{\sigma\gamma}\right) + \frac{1}{2}\kappa(\epsilon+3p) = 0. \end{aligned} \tag{12.42}$$

Now we use (12.29) and (12.27); the latter implies $\dot{u}^\alpha u_\alpha = 0$. We also use the definitions of θ, $\sigma^{\alpha\beta}$ and $\omega^{\alpha\beta}$ and (12.33). We obtain then in (12.42):

$$\begin{aligned} 0 &= \dot{\theta} - \dot{u}^\gamma{}_{;\gamma} + \sigma^{\sigma\gamma}u_{(\gamma;\sigma)} + \omega^{\sigma\gamma}u_{[\gamma;\sigma]} + \frac{1}{3}\theta u^\sigma{}_{;\sigma} + \frac{1}{2}\kappa(\epsilon+3p) \\ &= \dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^\gamma{}_{;\gamma} + \sigma^{\sigma\gamma}\left(\sigma_{\sigma\gamma} + \dot{u}_{(\sigma}u_{\gamma)} + \frac{1}{3}\theta h_{\sigma\gamma}\right) \\ &\qquad + \omega^{\sigma\gamma}\left(\omega_{\gamma\sigma} + \dot{u}_{[\gamma}u_{\sigma]}\right) + \frac{1}{2}\kappa(\epsilon+3p) \\ &= \dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^\gamma{}_{;\gamma} + \sigma^{\sigma\gamma}\sigma_{\sigma\gamma} - \omega^{\sigma\gamma}\omega_{\sigma\gamma} + \frac{1}{2}\kappa(\epsilon+3p). \end{aligned} \tag{12.43}$$

Finally, substituting (12.35) and (12.36) we obtain

$$\dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^\gamma{}_{;\gamma} + 2\left(\sigma^2 - \omega^2\right) + \frac{1}{2}\kappa(\epsilon+3p) = 0. \tag{12.44}$$

This equation was derived by Ehlers [101] in 1961. A subcase of it, corresponding to $p = 0$ and without the definitions of the shear and rotation, was derived by Raychaudhuri in 1955 [104]. Out of respect for the important idea, this equation is today called the **Raychaudhuri equation** (the name seems to have been introduced by Ellis [102]).

Taking the antisymmetric part of eq. (12.39), then its symmetric part and using in the second one the Raychaudhuri equation to eliminate $\dot{\theta}$, we obtain two other equations

- (a) The vorticity propagation equation

$$h^\gamma{}_\alpha h^\delta{}_\beta \dot{\omega}_{\gamma\delta} - h^\gamma{}_\alpha h^\delta{}_\beta \dot{u}_{[\gamma;\delta]} + 2\sigma_{\delta[\alpha}\omega^\delta{}_{\beta]} + \frac{2}{3}\theta\omega_{\alpha\beta} = 0; \tag{12.45}$$

[104] A. K. Raychaudhuri, *Phys. Rev.* **98**, 1123 (1955); **106**, 172 (1957). The first paper was reprinted in *Gen. Relativ. Gravit.* **32**, 749 (2000), with an editorial note by J. Earman and A. K. Raychaudhuri, *Gen. Relativ. Gravit.* **32**, 743 (2000) and author's (auto)biography by A. K. Raychaudhuri, *Gen. Relativ. Gravit.* **32**, 746 (2000).

- (b) The shear propagation equation

$$\begin{aligned} h^\gamma{}_\alpha h^\delta{}_\beta \dot{\sigma}_{\gamma\delta} &- h^\gamma{}_\alpha h^\delta{}_\beta \dot{u}_{(\gamma;\delta)} + \dot{u}_\alpha \dot{u}_\beta + \omega_{\alpha\gamma}\omega^\gamma{}_\beta + \sigma_{\alpha\gamma}\sigma^\gamma{}_\beta \\ &+ \frac{2}{3}\theta\sigma_{\alpha\beta} + \frac{1}{3}h_{\alpha\beta}\left[2\left(\omega^2 - \sigma^2\right) + \dot{u}^\gamma{}_{;\gamma}\right] + E_{\alpha\beta} = 0, \end{aligned} \tag{12.46}$$

where the quantity $E_{\alpha\beta}$ is the "electric part" of the Weyl tensor:

$$E_{\alpha\beta} \stackrel{\text{def}}{=} C_{\alpha\rho\beta\sigma}u^\rho u^\sigma = E_{\beta\alpha}. \tag{12.47}$$

In addition, the following three other equations hold:

$$\omega_{[\alpha\beta;\gamma]} + \dot{u}_{[\alpha;\gamma}u_{\beta]} + \dot{u}_{[\alpha}\omega_{\beta\gamma]} = 0, \tag{12.48}$$

$$h^\alpha{}_\beta\left(\omega^{\beta\gamma}{}_{;\gamma} - \sigma^{\beta\gamma}{}_{;\gamma} + \frac{2}{3}\theta^{;\beta}\right) - \left(\omega^\alpha{}_\beta + \sigma^\alpha{}_\beta\right)\dot{u}^\beta = 0, \tag{12.49}$$

$$2\dot{u}_{(\alpha}w_{\beta)} - \sqrt{-g}h^\gamma{}_\alpha h^\delta{}_\beta\left(\omega_{(\gamma}{}^{\mu;\nu} + \sigma_{(\gamma}{}^{\mu;\nu}\right)\epsilon_{\delta)\rho\mu\nu}u^\rho = H_{\alpha\beta}, \tag{12.50}$$

where $H_{\alpha\beta}$ is the "magnetic part" of the Weyl tensor:

$$H_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\gamma\mu\nu}C^{\mu\nu}{}_{\beta\delta}u^\gamma u^\delta = H_{\beta\alpha}. \tag{12.51}$$

Equation (12.48) is obtained by acting on (12.38) with the antisymmetrisation operator in the indices γ, δ and σ and by using $-R_{[\gamma|\rho|\delta\sigma]}u^\rho = u^\rho R_{\rho[\gamma\delta\sigma]} = 0$. Equation (12.49) is obtained by contracting (12.38) with $g^{\gamma\delta}h^{\alpha\sigma}$ and using (12.41). In order to obtain (12.50), one has to rewrite (12.38) in the form

$$-u_\delta{}^{;\mu\nu} + u_\delta{}^{;\nu\mu} = R^{\mu\nu}{}_{\delta\sigma}u^\sigma, \tag{12.52}$$

then act on both sides of (12.52) with the operator $\frac{1}{2}\sqrt{-g}\epsilon_{\gamma\rho\mu\nu}u^\rho h^\gamma{}_\alpha h^\delta{}_\beta$, and then symmetrise the result with respect to α and β.

Eqs. (12.44) – (12.50) are algebraically independent components of eq. (12.38).

A consequence of eqs. (12.44) – (12.50) is that assumptions made about the kinematical quantities can lead to restrictive results. For example, let us assume that $\dot{u}^\alpha = \sigma = \omega = 0$. Then (12.45) and (12.48) are fulfilled identically, (12.49) says that the expansion scalar may change only along the flow lines, while (12.46) and (12.50) imply that the Weyl tensor is zero – such metrics are called **conformally flat**. The family of solutions of the Einstein equations with perfect fluid source which are conformally flat was found by Stephani in 1967 [105] (see also [33]). They have the properties $\sigma = \omega = 0$, but in general $\dot{u}^\alpha \neq 0$. In the limit $\dot{u}^\alpha = 0$ they reduce to the Robertson–Walker metrics of Sec. 9.8 – which is a rather strong simplification. The full set of solutions of Einstein's equations with a perfect fluid source for which $\sigma = \omega = 0$ was found by Barnes in 1973 [103].

[105] H. Stephani *Commun. Math. Phys.* **4**, 137 (1967).

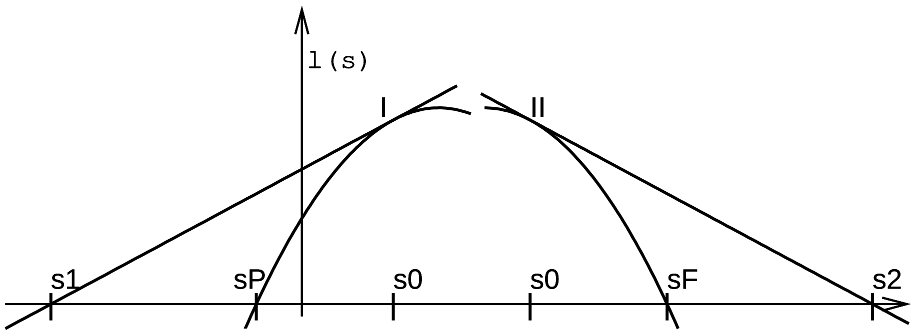

Figure 12.1: A function that is everywhere concave must go to zero either in the past (at $s_P > s_1$) or in the future (at $s_F < s_2$).

12.4 Singularities and singularity theorems

Let us define the function $\ell\,(x^\mu)$ as follows:

$$\frac{1}{\ell}\frac{\mathrm{d}\ell}{\mathrm{d}s} = \frac{1}{3}\theta, \tag{12.53}$$

where $\mathrm{d}/\mathrm{d}s = u^\rho \partial/\partial x^\rho$. It can be seen from (12.24), (12.19), (12.28) and (12.29) that with $\omega = \sigma = 0 = \dot{u}^\alpha$, the distance between the simultaneous positions of two particles obeys (12.53). In general, $\ell(x)$ has no direct physical interpretation and is just a convenient representation of θ. When $\omega = 0 = \dot{u}^\alpha$, the Raychaudhuri equation becomes

$$3\frac{\ddot{\ell}}{\ell} + 2\sigma^2 + \frac{1}{2}\kappa(\epsilon + 3p) = 0. \tag{12.54}$$

Since $\epsilon + 3p > 0$ for all kinds of matter known from laboratory, (12.54) shows that $\mathrm{d}^2\ell/\mathrm{d}s^2 < 0$. This means that the function $\ell(s)$ is concave in all its range – the *whole* curve $\ell(s)$ lies below its tangent at any point. There are two possibilities. If at present ($s = s_0$) $(\mathrm{d}\ell/\mathrm{d}s)(s_0) > 0$ (i.e. the fluid expands, curve I in Fig. 12.1), then at an instant s_p *in the past*, $s_1 < s_p < s_0$, ℓ *was zero*. If, however, $(\mathrm{d}\ell/\mathrm{d}s)(s_0) < 0$ at present (i.e. the fluid contracts, curve II in Fig. 12.1), then at an instant s_F *in the future*, $s_2 > s_F > s_0$, ℓ *will be zero*. Consequently, if $\dot{u}^\alpha = 0 = \omega$ then there exists such an instant, in the past or in the future, at which $\ell \to 0$. This implies, via (12.54), that $\epsilon + 3p \to \infty$ or $\sigma \to \infty$. Hence, every such portion of matter must have a singularity either in its future or in its past.

Note that we have proven here the existence of the singularity using the Einstein equations (12.41), but without invoking any specific solution of them.

By a similar analysis, with less restrictive assumptions, Penrose, Hawking and Ellis proved several theorems that imply that also quite general fluid configurations must contain singularities [106]. The theorems were said to show that general relativity cannot be the

[106] S. W. Hawking, G. F. R. Ellis, *The large-scale structure of spacetime*. Cambridge University Press 1973.

ultimate theory of space and time. In order to avoid the singularities, one would have to resort to a more general theory that would be capable of describing the quantum effects taking place at great densities of matter.

However, the singularity theorems are not as general as it was initially claimed. Several solutions of the Einstein equations that *do not* contain any singularities have been found by J. M. M. Senovilla and coworkers [107]. So far, they have not been related to any actual astrophysical situation, but their very existence proves that singularities are a consequence of the assumptions of the singularity theorems and not of relativity theory as such.

12.5 Exercises

1. Show that the quantity D defined in (12.17) is indeed zero.

Hint: Decompose the vectors $\delta\mathbf{x}$, $\delta\mathbf{y}$ and $\delta\mathbf{z}$ in the basis of the eigenvectors of σ_{ij}. Then use the equation $\sigma_{ij}n_j^{(a)} = \lambda_{(a)}n_i^{(a)}$ (no sum over a), where $n_j^{(a)}$ is the a-th eigenvector of σ_{ij}. Finally note that, in consequence of $\sigma_{ii} = 0$, the sum of eigenvalues of σ_{ij} is zero.

2. Verify eqs. (12.37).

[107] J. M. M. Senovilla, *Gen. Relativ. Gravit.* **30**, 701 (1998).