Relativistic Zel'dovich approximation and its applications

Jan Ostrowski

Nicolaus Copernicus University, Torun

General setting

- matter model: dust – no pressure
- 3+1 foliation
- synchronous gauge – no rotation
- Einstein–de Sitter background – no Λ
- we will use the principal scalar invariants of extrinsic curvature K_{ij} I, II, III and
- standard kinematical decomposition into: expansion rate $\Theta = -K^k_{\ k}$ and shear $\sigma^i_{\ j} = -K^i_{\ j} - \frac{1}{3}\Theta\delta^i_{\ j}$
- Buchert equations – solvable for special cases; Buchert equations + RZA –solvable for generic fields

Euler-Newton system

- Euler-Newton system (ENS): 2 evolution + 2 field equations:

Euler-Newton system

- Euler-Newton system (ENS): 2 evolution + 2 field equations:

$$\partial_t \vec{v} = -\left(\vec{v} \cdot \nabla\right) \vec{v} + \vec{g}$$

$$\partial_t \varrho = -\nabla \cdot \left(\varrho \vec{v}\right)$$

$$\nabla \times \vec{g} = \vec{0}$$

$$\nabla \cdot \vec{g} = -4\pi G \varrho$$

Euler-Newton system

- Euler-Newton system (ENS): 2 evolution + 2 field equations:

$$\partial_t \vec{v} = -\left(\vec{v} \cdot \nabla\right) \vec{v} + \vec{g}$$
$$\partial_t \varrho = -\nabla \cdot \left(\varrho \vec{v}\right)$$
$$\nabla \times \vec{g} = \vec{0}$$
$$\nabla \cdot \vec{g} = -4\pi G \varrho$$

- We introduce the trajectory field $\vec{f}(\vec{X}, t)$ and the Jacobian $J = \frac{1}{6}\epsilon_{ijk}\epsilon^{lmn} f^i{}_{|l} f^j{}_{|m} f^k{}_{|n}$ and implicitly solve the evolution equations by:

$$\vec{v} = \dot{\vec{f}} \;\; ; \;\; \vec{g} = \ddot{\vec{f}} \;\; ; \;\; \varrho = \frac{\mathring{\varrho}}{J} \;\; , \;\; J > 0$$

Lagrange-Newton system

- We define the functional determinant of three functions A, B, C:

$$\mathcal{J}(A,B,C) := \frac{\partial(A,B,C)}{\partial(X_1,X_2,X_3)} = \epsilon_{ijk} A_{|i} B_{|j} C_{|k}$$

Lagrange-Newton system

- We define the functional determinant of three functions A, B, C:

$$\mathcal{J}(A, B, C) := \frac{\partial(A, B, C)}{\partial(X_1, X_2, X_3)} = \epsilon_{ijk} A_{|i} B_{|j} C_{|k}$$

- The Lagrange-Newton System (LNS) takes the form:

Lagrange-Newton system

- We define the functional determinant of three functions A, B, C:

$$\mathcal{J}(A,B,C) := \frac{\partial(A,B,C)}{\partial(X_1,X_2,X_3)} = \epsilon_{ijk} A_{|i} B_{|j} C_{|k}$$

- The Lagrange-Newton System (LNS) takes the form:

$$\mathcal{J}(\ddot{f}^i, f^i, f^k) = 0$$
$$\mathcal{J}(\ddot{f}^1, f^2, f^3) + \text{cycl.} = -4\pi G \mathring{\varrho}$$

- This is equivalent to ENS provided that the trajectory field is unique and the Jacobian is greater than 0

First order scheme

- We decompose $\vec{f}$ into a homogeneous and isotropic background deformation $\vec{f}_H(\vec{X},t) = a(t)\vec{X}$ and an inhomogeneous deformation field $\vec{p}(\vec{X},t)$

$$\vec{f}(\vec{X},t) = a(t)\vec{X} + \vec{p}(\vec{X},t)$$

First order scheme

- We decompose $\vec{f}$ into a homogeneous and isotropic background deformation $\vec{f}_H(\vec{X},t) = a(t)\vec{X}$ and an inhomogeneous deformation field $\vec{p}(\vec{X},t)$

$$\vec{f}(\vec{X},t) = a(t)\vec{X} + \vec{p}(\vec{X},t)$$

- We introduce rescaled quantities $\vec{q} = \vec{F}(\vec{X},t) \equiv \vec{f}(\vec{X},t)/a(t)$ and $\vec{P}(\vec{X},t) \equiv \vec{p}(\vec{X},t)/a(t)$

First order scheme

- We decompose $\vec{f}$ into a homogeneous and isotropic background deformation $\vec{f}_H(\vec{X},t) = a(t)\vec{X}$ and an inhomogeneous deformation field $\vec{p}(\vec{X},t)$

$$\vec{f}(\vec{X},t) = a(t)\vec{X} + \vec{p}(\vec{X},t)$$

- We introduce rescaled quantities $\vec{q} = \vec{F}(\vec{X},t) \equiv \vec{f}(\vec{X},t)/a(t)$ and $\vec{P}(\vec{X},t) \equiv \vec{p}(\vec{X},t)/a(t)$

- First-order equations for the inhomogeneous part are:

$$a^2\nabla_0 \times \ddot{\vec{p}} - \ddot{a}a\nabla_0 \times \vec{p} = \vec{0}$$

$$a^2\nabla_0 \cdot \ddot{\vec{p}} + (2\ddot{a}a)\,\nabla_0 \cdot \vec{p} - 4\pi G\,(\mathring{\varrho} - \mathring{\varrho}_H)$$

- Splitting scaled perturbation field into transverse (divergence-free) and longitudinal (curl-free) parts we obtain:

- Splitting scaled perturbation field into transverse (divergence-free) and longitudinal (curl-free) parts we obtain:

$$\ddot{\vec{P}}^T + 2H\dot{\vec{P}}^T = \vec{0}$$

$$\ddot{\vec{P}}^L + 2H\dot{\vec{P}}^L - 4\pi G \varrho_H \vec{P}^L = \frac{1}{a^3}\vec{W}(\vec{X})$$

where $\vec{W}(\vec{X}) \equiv \ddot{\vec{P}}^L(\vec{X}, t_0) + 2H(t_0)\dot{\vec{P}}^L(\vec{X}, t_0)$ is the initial peculiar-acceleration field.

- Splitting scaled perturbation field into transverse (divergence-free) and longitudinal (curl-free) parts we obtain:

$$\ddot{\vec{P}}^T + 2H\dot{\vec{P}}^T = \vec{0}$$
$$\ddot{\vec{P}}^L + 2H\dot{\vec{P}}^L - 4\pi G\varrho_H \vec{P}^L = \frac{1}{a^3}\vec{W}(\vec{X})$$

where $\vec{W}(\vec{X}) \equiv \ddot{\vec{P}}^L(\vec{X}, t_0) + 2H(t_0)\dot{\vec{P}}^L(\vec{X}, t_0)$ is the initial peculiar-acceleration field.

- Special case of first-order solution - Zel'dovich approximation:

$$\vec{u}(\vec{X}, t) = \vec{w}(\vec{X}, t)\, t \ ; \ \ t = t_0 \ \ ,$$

where

$$\vec{u} = a\dot{\vec{P}} \, , \quad \text{and} \quad \vec{w} = \dot{\vec{u}} + H\vec{u} = 2\dot{a}\dot{\vec{P}} + a\ddot{\vec{P}} \ \ .$$

Basic scheme

- We employ the ADM equations and express them with a single variable i.e. the cartan co-frame η^a_i (*M.Kasai, PRD 52, 5605 (1995); T.Buchert and M.Ostermann, PRD 86, 023520 (2012) arXiv:1203.6263*

Basic scheme

- We employ the ADM equations and express them with a single variable i.e. the cartan co-frame η^a_i (*M.Kasai, PRD 52, 5605 (1995); T.Buchert and M.Ostermann, PRD 86, 023520 (2012) arXiv:1203.6263*

- Whenever possible we reduce the constraint equations to constraints on the initial hypersurface together with time dependent functions

Basic scheme

- We employ the ADM equations and express them with a single variable i.e. the cartan co-frame η^a_i *(M.Kasai, PRD 52, 5605 (1995); T.Buchert and M.Ostermann, PRD 86, 023520 (2012) arXiv:1203.6263*
- Whenever possible we reduce the constraint equations to constraints on the initial hypersurface together with time dependent functions
- We derive the general first order solutions for the dynamical variable

Basic scheme

- We employ the ADM equations and express them with a single variable i.e. the cartan co-frame η^a_i (*M.Kasai, PRD 52, 5605 (1995); T.Buchert and M.Ostermann, PRD 86, 023520 (2012) arXiv:1203.6263*
- Whenever possible we reduce the constraint equations to constraints on the initial hypersurface together with time dependent functions
- We derive the general first order solutions for the dynamical variable
- We use the perturbed deformation to functionally evaluate the quantities of interest e.g. density, curvature

Lagrange-Einstein system

- Lagrange-Einstein System in ADM formulation

$$\delta_{ab}\ddot{\eta}^{a}{}_{[i}\eta^{b}{}_{j]} = 0$$

$$\frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^{a}{}_{i}\eta^{b}{}_{k}\eta^{c}{}_{l} = \Lambda J - 4\pi G\mathring{J}\mathring{\varrho}$$

$$\left(\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^{a}{}_{j}\eta^{b}{}_{k}\eta^{c}{}_{l}\right)_{\|i} = \left(\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^{a}{}_{i}\eta^{b}{}_{k}\eta^{c}{}_{l}\right)_{\|j}$$

$$\epsilon_{abc}\epsilon^{mkl}\dot{\eta}^{a}{}_{m}\dot{\eta}^{b}{}_{k}\eta^{c}{}_{l} = 16\pi G\mathring{J}\mathring{\varrho} - JR$$

$$\frac{1}{2}\left(\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^{a}{}_{j}\eta^{b}{}_{k}\eta^{c}{}_{l} - \frac{1}{3}\epsilon_{abc}\epsilon^{mkl}\ddot{\eta}^{a}{}_{m}\eta^{b}{}_{k}\eta^{c}{}_{l}\delta^{i}{}_{j}\right)$$
$$+\left(\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^{a}{}_{j}\dot{\eta}^{b}{}_{k}\eta^{c}{}_{l} - \frac{1}{3}\epsilon_{abc}\epsilon^{mkl}\dot{\eta}^{a}{}_{m}\dot{\eta}^{b}{}_{k}\eta^{c}{}_{l}\delta^{i}{}_{j}\right)$$
$$= -J\tau^{i}_{j}$$

First order scheme

- We decompose the Cartan coframe into a flat, homogeneous, isotropic background and an inhomogeneous deviation. We also define the peculiar coframe

First order scheme

- We decompose the Cartan coframe into a flat, homogeneous, isotropic background and an inhomogeneous deviation. We also define the peculiar coframe

$$\eta^a{}_i = a(t)\left[\delta^a{}_i + P^a{}_i(X,t)\right] \, , \qquad \tilde{\eta}^a{}_i \equiv \frac{1}{a}\eta^a{}_i$$

First order scheme

- We decompose the Cartan coframe into a flat, homogeneous, isotropic background and an inhomogeneous deviation. We also define the peculiar coframe

$$\eta^a{}_i = a(t)\left[\delta^a{}_i + P^a{}_i(X,t)\right] , \qquad \tilde{\eta}^a{}_i \equiv \frac{1}{a}\eta^a{}_i$$

- We derive the linearized evolution equations and divide them into trace and trace-free part

First order scheme

- We decompose the Cartan coframe into a flat, homogeneous, isotropic background and an inhomogeneous deviation. We also define the peculiar coframe

$$\eta^a{}_i = a(t)\left[\delta^a{}_i + P^a{}_i(X,t)\right] , \qquad \tilde{\eta}^a{}_i \equiv \frac{1}{a}\eta^a{}_i$$

- We derive the linearized evolution equations and divide them into trace and trace-free part

$$\ddot{P} + 3H\dot{P} = \frac{1}{a^2}\big(\ddot{P}(t_0) + 3H(t_0)\dot{P}(t_0)\big)$$

First order scheme

- We decompose the Cartan coframe into a flat, homogeneous, isotropic background and an inhomogeneous deviation. We also define the peculiar coframe

$$\eta^a{}_i = a(t)\left[\delta^a{}_i + P^a{}_i(X,t)\right] , \qquad \tilde{\eta}^a{}_i \equiv \frac{1}{a}\eta^a{}_i$$

- We derive the linearized evolution equations and divide them into trace and trace-free part

$$\ddot{P} + 3H\dot{P} = \frac{1}{a^2}\left(\ddot{P}(t_0) + 3H(t_0)\dot{P}(t_0)\right)$$

$$\ddot{\Pi}^i{}_j + 3H\dot{\Pi}^i{}_j = -{}^{(1)}\tau^i{}_j$$

where $\Pi^i{}_j \equiv P^i{}_j - \frac{1}{3}P\delta^i{}_j$.

General first-order solution

- We separate the time and spatial derivatives and make the ansatz

$$P^a{}_i(X,t) = {}^0Q^a{}_i(X) + q_1(t)\,{}^1Q^a{}_i(X) + q_2(t)\,{}^2Q^a{}_i(X)$$

where the time functions $q_{1/2}(t)$ are the two solutions of:

$$\ddot{q} + 2\frac{\dot{a}}{a}\dot{q} + \left(3\frac{\ddot{a}}{a} - \Lambda\right)(q + q(t_0)) = 0\,.$$

General first-order solution

- We separate the time and spatial derivatives and make the ansatz

$$P^a{}_i(X,t) = {}^0Q^a{}_i(X) + q_1(t)\,{}^1Q^a{}_i(X) + q_2(t)\,{}^2Q^a{}_i(X)$$

where the time functions $q_{1/2}(t)$ are the two solutions of:

$$\ddot{q} + 2\frac{\dot{a}}{a}\dot{q} + \left(3\frac{\ddot{a}}{a} - \Lambda\right)(q + q(t_0)) = 0\,.$$

- With the ansatz and its time derivatives we find

$$\begin{aligned} {}^1Q^a{}_i &= +\frac{\dot{q}_2(t_0)\ddot{P}^a{}_i(t_0) - \ddot{q}_2(t_0)\dot{P}^a{}_i(t_0)}{\ddot{q}_1(t_0)\dot{q}_2(t_0) - \dot{q}_1(t_0)\ddot{q}_2(t_0)}\,, \\ {}^2Q^a{}_i &= -\frac{\dot{q}_1(t_0)\ddot{P}^a{}_i(t_0) - \ddot{q}_1(t_0)\dot{P}^a{}_i(t_0)}{\ddot{q}_1(t_0)\dot{q}_2(t_0) - \dot{q}_1(t_0)\ddot{q}_2(t_0)} \end{aligned}$$

- … together with

$$ {}^{0}Q^{a}{}_{i} = P^{a}{}_{i}(t_0) - q_1(t_0)\,{}^{1}Q^{a}{}_{i} - q_2(t_0)\,{}^{2}Q^{a}{}_{i} $$

- ... together with

$$ {}^{0}Q^{a}{}_{i} = P^{a}{}_{i}(t_0) - q_1(t_0)\,{}^{1}Q^{a}{}_{i} - q_2(t_0)\,{}^{2}Q^{a}{}_{i} $$

- Finally we obtain an expression for the first-order peculiar coframe:

$$ {}^{(1)}\tilde{\eta}^{a}{}_{i} = \mathring{\eta}^{a}{}_{i} + \big(q_1(t) - q_1(t_0)\big)\,{}^{1}Q^{a}{}_{i} + \big(q_2(t) - q_2(t_0)\big)\,{}^{2}Q^{a}{}_{i} $$

where $\mathring{\eta}^{a}{}_{i} \equiv \delta^{a}{}_{i} + P^{a}{}_{i}(t_0)$ is the coframe at the initial time.

- … together with

$$ {}^{0}Q^{a}{}_{i} = P^{a}{}_{i}(t_0) - q_1(t_0)\,{}^{1}Q^{a}{}_{i} - q_2(t_0)\,{}^{2}Q^{a}{}_{i} $$

- Finally we obtain an expression for the first-order peculiar coframe:

$$ {}^{(1)}\tilde{\eta}^{a}{}_{i} = \mathring{\eta}^{a}{}_{i} + \big(q_1(t) - q_1(t_0)\big)\,{}^{1}Q^{a}{}_{i} + \big(q_2(t) - q_2(t_0)\big)\,{}^{2}Q^{a}{}_{i} $$

where $\mathring{\eta}^{a}{}_{i} \equiv \delta^{a}{}_{i} + P^{a}_{i}(t_0)$ is the coframe at the initial time.
- Additionally, we define some useful peculiar-quantities $u^{a}{}_{i}$ and $w^{a}{}_{i}$:

$$ \begin{aligned} \dot{\eta}^{a}{}_{i} &= H\eta^{a}{}_{i} + u^{a}{}_{i}\,, \quad u^{a}{}_{i} \equiv a\dot{P}^{a}_{i}\,, \\ \ddot{\eta}^{a}{}_{i} &= \frac{\ddot{a}}{a}\eta^{a}{}_{i} + w^{a}{}_{i}\,, \quad w^{a}{}_{i} \equiv a\ddot{P}^{a}_{i} + 2\dot{a}\dot{P}^{a}_{i} \end{aligned} $$

Relativistic Zel'dovich Approximation

- In analogy to the Newtonian investigation we restrict ourselves to the trace part and we impose the following slaving conditions:

$$ {}^{2}Q^{a}{}_{i}(X) = 0 \qquad w^{a}{}_{i} = \left(2H + \frac{\ddot{q}_1}{\dot{q}_1}\right) u^{a}{}_{i} $$

Relativistic Zel'dovich Approximation

- In analogy to the Newtonian investigation we restrict ourselves to the trace part and we impose the following slaving conditions:

$$ {}^{2}Q^{a}{}_{i}(X) = 0 \qquad w^{a}{}_{i} = \left(2H + \frac{\ddot{q}_1}{\dot{q}_1}\right) u^{a}{}_{i} $$

resulting in:

$$ {}^{1}Q^{a}{}_{i}(X) = \frac{1}{\dot{q}_1(t_0)} \dot{P}^{a}{}_{i}(X, t_0) $$

Relativistic Zel'dovich Approximation

- In analogy to the Newtonian investigation we restrict ourselves to the trace part and we impose the following slaving conditions:

$$^{2}Q^{a}{}_{i}(X) = 0 \qquad w^{a}{}_{i} = \left(2H + \frac{\ddot{q}_1}{\dot{q}_1}\right)u^{a}{}_{i}$$

resulting in:

$$^{1}Q^{a}{}_{i}(X) = \frac{1}{\dot{q}_1(t_0)}\dot{P}^{a}{}_{i}(X, t_0)$$

- Thus, we obtain the expression for the peculiar coframe:

$$^{\mathsf{RZA}}\tilde{\eta}^{a}{}_{i}(X,t) = \delta^{a}{}_{i} + P^{a}{}_{i}(X,t_0) + \xi(t)\dot{P}^{a}{}_{i}(X,t_0)$$
$$\xi(t) \equiv (q_1(t) - q_1(t_0))/\dot{q}_1(t_0)$$

Averaged equations I

- We define the domain dependent scale factor

$$a_{\mathcal{D}}(t) := \left(\frac{V_{\mathcal{D}}(t)}{V_{\mathcal{D}_{\mathbf{i}}}} \right)^{1/3}$$

where the volume of the domain is given by:

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \mathrm{d}\mu_{\mathrm{g}}$$

Averaged equations I

- We define the domain dependent scale factor

$$a_{\mathcal{D}}(t) := \left(\frac{V_{\mathcal{D}}(t)}{V_{\mathcal{D}_\mathrm{i}}}\right)^{1/3}$$

where the volume of the domain is given by:

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \mathrm{d}\mu_\mathrm{g}$$

- We apply the following commutation rule to the Raychaudhuri and Hamilton equations

$$\partial_t \langle \Psi(t, X^k) \rangle_{\mathcal{D}} - \langle \partial_t \Psi(t, X^k) \rangle_{\mathcal{D}} = \langle \Theta \Psi \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}} \langle \Psi \rangle_{\mathcal{D}}$$

- We obtain the generalised Friedmann equations for inhomogeneous fluids:

  $\rightarrow$ the averaged Raychaudhuri equation:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M_{\mathcal{D}_{\mathbf{i}}}}{V_{\mathcal{D}_{\mathbf{i}}} a_{\mathcal{D}}^3} = \mathcal{Q}_{\mathcal{D}}$$

- We obtain the generalised Friedmann equations for inhomogeneous fluids:
  - $\rightarrow$ the averaged Raychaudhuri equation:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{M_{\mathcal{D}_{\mathbf{i}}}}{V_{\mathcal{D}_{\mathbf{i}}} a_{\mathcal{D}}^3} = \mathcal{Q}_{\mathcal{D}}$$

  - $\rightarrow$ the averaged Hamiltonian constraint

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - \frac{8\pi G}{3}\frac{M_{\mathcal{D}_{\mathbf{i}}}}{V_{\mathcal{D}_{\mathbf{i}}} a_{\mathcal{D}}^3} + \frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6} \; ,$$

- We obtain the generalised Friedmann equations for inhomogeneous fluids:
  - $\rightarrow$ the averaged Raychaudhuri equation:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{M_{\mathcal{D}_{\mathbf{i}}}}{V_{\mathcal{D}_{\mathbf{i}}} a_{\mathcal{D}}^3} = \mathcal{Q}_{\mathcal{D}}$$

  - $\rightarrow$ the averaged Hamiltonian constraint

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - \frac{8\pi G}{3}\frac{M_{\mathcal{D}_{\mathbf{i}}}}{V_{\mathcal{D}_{\mathbf{i}}} a_{\mathcal{D}}^3} + \frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6} \ ,$$

where the kinematical backreaction term is given by:

$$\mathcal{Q}_{\mathcal{D}} = 2\langle\mathrm{II}\rangle_{\mathcal{D}} - \frac{2}{3}\langle\mathrm{I}\rangle_{\mathcal{D}}^2 = \frac{2}{3}\langle(\Theta - \langle\Theta\rangle_{\mathcal{D}})^2\rangle_{\mathcal{D}} - 2\langle\sigma^2\rangle_{\mathcal{D}}$$

Kinematical backreaction

With the perturbed co-frame ${}^{\text{RZA}}\tilde{\eta}^a_{\ i}(X,t) = \delta^a_{\ i} + P^a_{\ i}(X,t_0) + \xi(t)\dot{P}^a_{\ i}(X,t_0)$ the backreaction takes the form (*T.Buchert et al., PRD 87, 123503 (2013), arXiv: 1303.6193*):

$$ {}^{\mathrm{RZA}}\mathcal{Q}_{\mathcal{D}} \;=\; \frac{\dot{\xi}^2\left(\gamma_1 + \xi\gamma_2 + \xi^2\gamma_3\right)}{\left(1 + \xi\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}} + \xi^2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} + \xi^3\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}}\right)^2} $$

with:

(1) $\gamma_1 := 2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}^2$

(2) $\gamma_2 := 6\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}}\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}$

(3) $\gamma_3 := 2\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}}^2$

Intrinsic curvature

$$ {}^{\mathrm{RZA}}\mathcal{R}_{\mathcal{D}} = \frac{\dot{\xi}^2\left(\tilde{\gamma}_1 + \xi\tilde{\gamma}_2 + \xi^2\tilde{\gamma}_3\right)}{1 + \xi\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}} + \xi^2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} + \xi^3\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}}} $$

with:

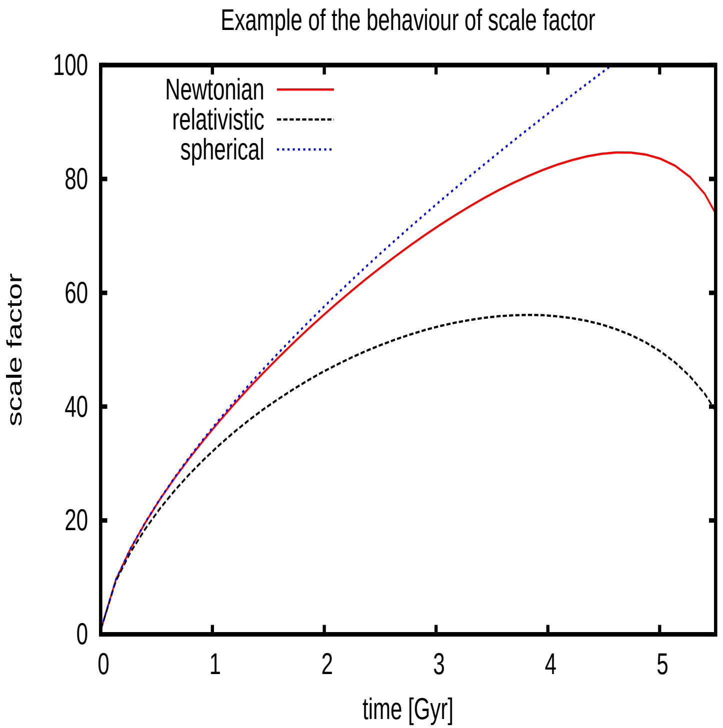
$$ \tilde{\gamma}_1 = -2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} - 12\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}\frac{H}{\dot{\xi}} - 4\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}\frac{\ddot{\xi}}{\dot{\xi}^2} $$

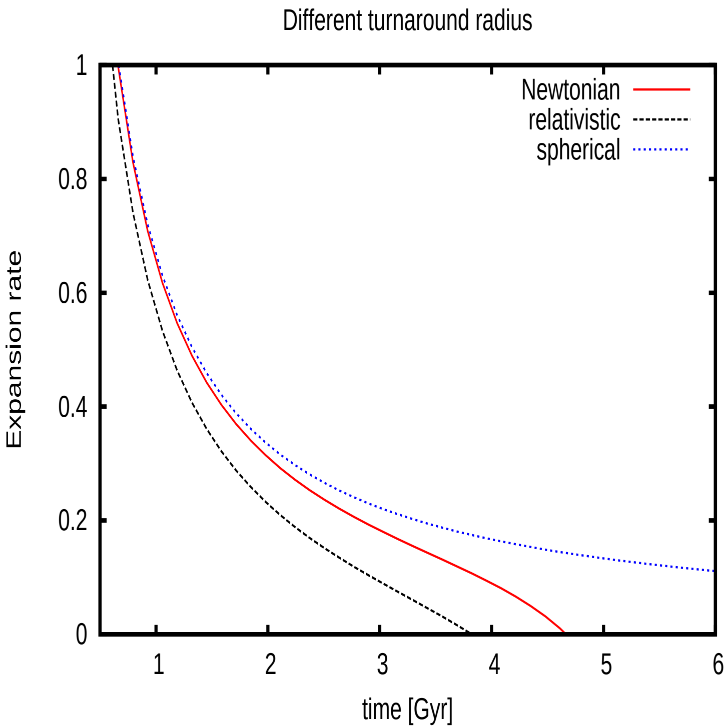
$$ \tilde{\gamma}_2 = -6\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}} - 24\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}}\frac{H}{\dot{\xi}} - 8\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}}\frac{\ddot{\xi}}{\dot{\xi}^2} $$

$$ \tilde{\gamma}_3 = -36\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}}\frac{H}{\dot{\xi}} - 12\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}}\frac{\ddot{\xi}}{\dot{\xi}^2} $$

Scale factors, expansion rates

Examples of the intrinsic curvature effects on scale factor and expansion rate of collapsing domain





General setting
Euler-Newton system
Lagrange-Newton system
First order scheme

Basic scheme
Lagrange-Einstein system
First order scheme
General first-order solution

Relativistic Zel'dovich Approximation
Averaged equations I

Kinematical backreaction
Intrinsic curvature
Scale factors, expansion rates

On the Green and Wald formalism
Brief history of the debate on Green and Wald formalism
Assumptions (Following arXiv:1011.4920v2 [gr-qc])
Weak limit, Green and Wald equations
Green and Wald theorems

On the Green and Wald formalism

Brief history of the debate on Green and Wald formalism

- Ishibashi and Wald: 'Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?'(arXiv:gr-qc/0509108) – negligible backreaction

Brief history of the debate on Green and Wald formalism

- Ishibashi and Wald: 'Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?'(arXiv:gr-qc/0509108) – negligible backreaction
- Introduction of the formalism by Green and Wald: 'A new framework for analyzing the effects of small scale inhomogeneities in cosmology' (arXiv: 1011.4920[gr-qc]) – backreaction can be large but it's traceless

Brief history of the debate on Green and Wald formalism

- Ishibashi and Wald: 'Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?'(arXiv:gr-qc/0509108) – negligible backreaction
- Introduction of the formalism by Green and Wald: 'A new framework for analyzing the effects of small scale inhomogeneities in cosmology' (arXiv: 1011.4920[gr-qc]) – backreaction can be large but it's traceless
- Examples by Green and Wald: 'Examples of backreaction of small scale inhomogeneities in cosmology' (arXiv:1304.2318[gr-qc])

Brief history of the debate on Green and Wald formalism

- Ishibashi and Wald: 'Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?'(arXiv:gr-qc/0509108) – negligible backreaction
- Introduction of the formalism by Green and Wald: 'A new framework for analyzing the effects of small scale inhomogeneities in cosmology' (arXiv: 1011.4920[gr-qc]) – backreaction can be large but it's traceless
- Examples by Green and Wald: 'Examples of backreaction of small scale inhomogeneities in cosmology' (arXiv:1304.2318[gr-qc])
- Overview paper by Green and Wald: 'How well is our universe described by an FLRW model?' (arXiv:1407.8084[gr-qc]

Brief history of the debate on Green and Wald formalism

- Ishibashi and Wald: 'Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?'(arXiv:gr-qc/0509108) – negligible backreaction
- Introduction of the formalism by Green and Wald: 'A new framework for analyzing the effects of small scale inhomogeneities in cosmology' (arXiv: 1011.4920[gr-qc]) – backreaction can be large but it's traceless
- Examples by Green and Wald: 'Examples of backreaction of small scale inhomogeneities in cosmology' (arXiv:1304.2318[gr-qc])
- Overview paper by Green and Wald: 'How well is our universe described by an FLRW model?' (arXiv:1407.8084[gr-qc]
- Rebuttal paper by Buchert et al.: 'Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?' (arXiv:1505.07800[gr-qc])

Brief history of the debate on Green and Wald formalism

- Ishibashi and Wald: 'Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?'(arXiv:gr-qc/0509108) – negligible backreaction
- Introduction of the formalism by Green and Wald: 'A new framework for analyzing the effects of small scale inhomogeneities in cosmology' (arXiv: 1011.4920[gr-qc]) – backreaction can be large but it's traceless
- Examples by Green and Wald: 'Examples of backreaction of small scale inhomogeneities in cosmology' (arXiv:1304.2318[gr-qc])
- Overview paper by Green and Wald: 'How well is our universe described by an FLRW model?' (arXiv:1407.8084[gr-qc]
- Rebuttal paper by Buchert et al.: 'Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?' (arXiv:1505.07800[gr-qc])
- Response to rebuttal, by Green and Wald: 'Comments on Backreaction' (arXiv: 1506.06452[gr-qc])

Assumptions (Following arXiv:1011.4920v2 [gr-qc])

- For all $\lambda > 0$ the metric $g_{ab}(\lambda, x)$ satisfies:

$$G_{ab}(g(\lambda, x)) + \Lambda g_{ab}(\lambda, x) = 8\pi T_{ab}(\lambda),$$

  where $T_{ab}(\lambda)$ obeys the weak energy condition

Assumptions (Following arXiv:1011.4920v2 [gr-qc])

- For all $\lambda > 0$ the metric $g_{ab}(\lambda, x)$ satisfies:

$$G_{ab}(g(\lambda, x)) + \Lambda g_{ab}(\lambda, x) = 8\pi T_{ab}(\lambda),$$

  where $T_{ab}(\lambda)$ obeys the weak energy condition

- There exists a smooth function $C_1(x)$ on M such that:

$$|h_{ab}(\lambda, x)| \leq \lambda C_1(x) \;\; ; \;\; h_{ab}(\lambda, x) = g_{ab}(\lambda, x) - g_{ab}(0, x).$$

Assumptions (Following arXiv:1011.4920v2 [gr-qc])

- For all $\lambda > 0$ the metric $g_{ab}(\lambda, x)$ satisfies:

$$G_{ab}(g(\lambda, x)) + \Lambda g_{ab}(\lambda, x) = 8\pi T_{ab}(\lambda),$$

where $T_{ab}(\lambda)$ obeys the weak energy condition

- There exists a smooth function $C_1(x)$ on M such that:

$$|h_{ab}(\lambda, x)| \leq \lambda C_1(x) \;\; ; \;\; h_{ab}(\lambda, x) = g_{ab}(\lambda, x) - g_{ab}(0, x).$$

- There exists a smooth function $C_2(x)$ on M such that: $|\nabla_c h_{ab}(\lambda, x)| \leq C_2(x)$.

Assumptions (Following arXiv:1011.4920v2 [gr-qc])

- For all $\lambda > 0$ the metric $g_{ab}(\lambda, x)$ satisfies:

$$G_{ab}(g(\lambda, x)) + \Lambda g_{ab}(\lambda, x) = 8\pi T_{ab}(\lambda),$$

where $T_{ab}(\lambda)$ obeys the weak energy condition

- There exists a smooth function $C_1(x)$ on M such that:

$$|h_{ab}(\lambda, x)| \leq \lambda C_1(x) \;\; ; \;\; h_{ab}(\lambda, x) = g_{ab}(\lambda, x) - g_{ab}(0, x).$$

- There exists a smooth function $C_2(x)$ on M such that: $|\nabla_c h_{ab}(\lambda, x)| \leq C_2(x)$.

- There exists a smooth tensor field μ_{abcdef} on M such that:

$$\underset{\lambda \searrow 0}{\text{w--lim}}(\nabla_a h_{cd}(\lambda, x) \nabla_b h_{ef}(\lambda, x)) = \mu_{abcdef}.$$

Weak limit, Green and Wald equations

- We say that $A_{a_1\ldots a_n}(\lambda)$ converges weakly to $B_{a_1\ldots a_n}$ i.e. w–$\lim_{\lambda\searrow 0} A_{a_1\ldots a_n}(\lambda) = B_{a_1\ldots a_n}$ when for all $f^{a_1\cdots a_n}$ of compact support:

$$\lim_{\lambda\searrow 0}\int f^{a_1\ldots a_n} B_{a_1\ldots a_n}(\lambda) = \int f^{a_1\ldots a_n} A_{a_1\ldots a_n}.$$

Weak limit, Green and Wald equations

- We say that $A_{a_1 \dots a_n}(\lambda)$ converges weakly to $B_{a_1 \dots a_n}$ i.e. $\text{w–lim}_{\lambda \searrow 0} A_{a_1 \dots a_n}(\lambda) = B_{a_1 \dots a_n}$ when for all $f^{a_1 \cdots a_n}$ of compact support:

$$\lim_{\lambda \searrow 0} \int f^{a_1 \dots a_n} B_{a_1 \dots a_n}(\lambda) = \int f^{a_1 \dots a_n} A_{a_1 \dots a_n} .$$

- Green and Wald equations for the background metric $g_{ab}(0,x)$ then reads:

$$\int f^{ab} G_{ab}(g_{ab}(0,x)) + \Lambda g_{ab}(0,x) = 8\pi \, \underset{\lambda \searrow 0}{\text{w–lim}} \left(T_{ab}(\lambda) + t_{ab}(\lambda) \right),$$

Weak limit, Green and Wald equations

- We say that $A_{a_1 \dots a_n}(\lambda)$ converges weakly to $B_{a_1 \dots a_n}$ i.e. w–lim$_{\lambda \searrow 0} A_{a_1 \dots a_n}(\lambda) = B_{a_1 \dots a_n}$ when for all $f^{a_1 \cdots a_n}$ of compact support:

$$\lim_{\lambda \searrow 0} \int f^{a_1 \dots a_n} B_{a_1 \dots a_n}(\lambda) = \int f^{a_1 \dots a_n} A_{a_1 \dots a_n} .$$

- Green and Wald equations for the background metric $g_{ab}(0, x)$ then reads:

$$\int f^{ab} G_{ab}(g_{ab}(0, x)) + \Lambda g_{ab}(0, x) = 8\pi \operatorname*{w\text{–}lim}_{\lambda \searrow 0} \left(T_{ab}(\lambda) + t_{ab}(\lambda) \right) ,$$

where:

$$t_{ab}(\lambda) = 2\nabla_{[a} C^e{}_{e]b} - 2C^f{}_{b[a} C^e{}_{e]f} - g_{ab}(\lambda) g^{cd}(\lambda) \nabla_{[c} C^e{}_{e]d} + g_{ab}(\lambda) g^{cd}(\lambda) C^f{}_{d[c} C^e{}_{e]f} ,$$

and

$$C^c{}_{ab} = \frac{1}{2} g^{cd}(\lambda) \left\{ \nabla_a g_{bd}(\lambda) + \nabla_b g_{ad}(\lambda) - \nabla_d g_{ab}(\lambda) \right\} .$$

Green and Wald theorems

- Green and Wald equation can be written symbolically:

$$G_{ab}(g^{(0)}) + \Lambda g_{ab}^{(0)} = 8\pi T_{ab}^{(0)} + 8\pi t_{ab}^{(0)}$$

Green and Wald theorems

- Green and Wald equation can be written symbolically:

$$G_{ab}(g^{(0)}) + \Lambda g_{ab}^{(0)} = 8\pi T_{ab}^{(0)} + 8\pi t_{ab}^{(0)}$$

- Green and Wald theorems concern the features of 'effective' stress-energy tensor: $t_{ab}^{(0)}$:

Green and Wald theorems

- Green and Wald equation can be written symbolically:

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}$$

- Green and Wald theorems concern the features of 'effective' stress-energy tensor: $t^{(0)}_{ab}$:

$\rightarrow$ $t^{(0)}_{ab}$ is traceless i.e. $t^{(0)a}{}_{a} = 0$

Green and Wald theorems

- Green and Wald equation can be written symbolically:

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}$$

- Green and Wald theorems concern the features of 'effective' stress-energy tensor: $t^{(0)}_{ab}$:

$\rightarrow$ $t^{(0)}_{ab}$ is traceless i.e. $t^{(0)a}{}_{a} = 0$

$\rightarrow$ $t^{(0)}_{ab}$ obeys the weak energy condition i.e. $t^{(0)}_{ab} t^a t^b \geq 0$

Green and Wald theorems

- Green and Wald equation can be written symbolically:

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}$$

- Green and Wald theorems concern the features of 'effective' stress-energy tensor: $t^{(0)}_{ab}$:

  $\rightarrow$ $t^{(0)}_{ab}$ is traceless i.e. $t^{(0)a}{}_{a} = 0$

  $\rightarrow$ $t^{(0)}_{ab}$ obeys the weak energy condition i.e. $t^{(0)}_{ab} t^a t^b \geq 0$

- To put it in words: $t^{(0)}_{ab}$ can not mimic the dark energy.

Inapplicability

- Green and Wald formalism does not apply to the situations when:

Inapplicability

- Green and Wald formalism does not apply to the situations when:

  $\rightarrow$ the actual metric is far from FLRW (e.g. LTB metric)

Inapplicability

- Green and Wald formalism does not apply to the situations when:
  - → the actual metric is far from FLRW (e.g. LTB metric)
  - → one wishes to construct an effective metric (or other effective quantities) via some averaging or smoothing procedure (it does not apply to e.g. Buchert formalism and many others in the literature as explicitly stated by Green and Wald in 'Comments on backreaction')

Inapplicability

- Green and Wald formalism does not apply to the situations when:
  - → the actual metric is far from FLRW (e.g. LTB metric)
  - → one wishes to construct an effective metric (or other effective quantities) via some averaging or smoothing procedure (it does not apply to e.g. Buchert formalism and many others in the literature as explicitly stated by Green and Wald in 'Comments on backreaction')
- What is then Green and Wald formalism' domain of application?

Inapplicability

- Green and Wald formalism does not apply to the situations when:
  - → the actual metric is far from FLRW (e.g. LTB metric)
  - → one wishes to construct an effective metric (or other effective quantities) via some averaging or smoothing procedure (it does not apply to e.g. Buchert formalism and many others in the literature as explicitly stated by Green and Wald in 'Comments on backreaction')
- What is then Green and Wald formalism' domain of application?
  - → backreaction with no backreaction

Inapplicability

- Green and Wald formalism does not apply to the situations when:
  - $\rightarrow$ the actual metric is far from FLRW (e.g. LTB metric)
  - $\rightarrow$ one wishes to construct an effective metric (or other effective quantities) via some averaging or smoothing procedure (it does not apply to e.g. Buchert formalism and many others in the literature as explicitly stated by Green and Wald in 'Comments on backreaction')
- What is then Green and Wald formalism' domain of application?
  - $\rightarrow$ backreaction with no backreaction
  - $\rightarrow$ averaging without averaging

Inapplicability

- Green and Wald formalism does not apply to the situations when:

  → the actual metric is far from FLRW (e.g. LTB metric)

  → one wishes to construct an effective metric (or other effective quantities) via some averaging or smoothing procedure (it does not apply to e.g. Buchert formalism and many others in the literature as explicitly stated by Green and Wald in 'Comments on backreaction')

- What is then Green and Wald formalism' domain of application?

  → backreaction with no backreaction
  → averaging without averaging
  → uniform vs non-uniform convergence

Problems with interpretation

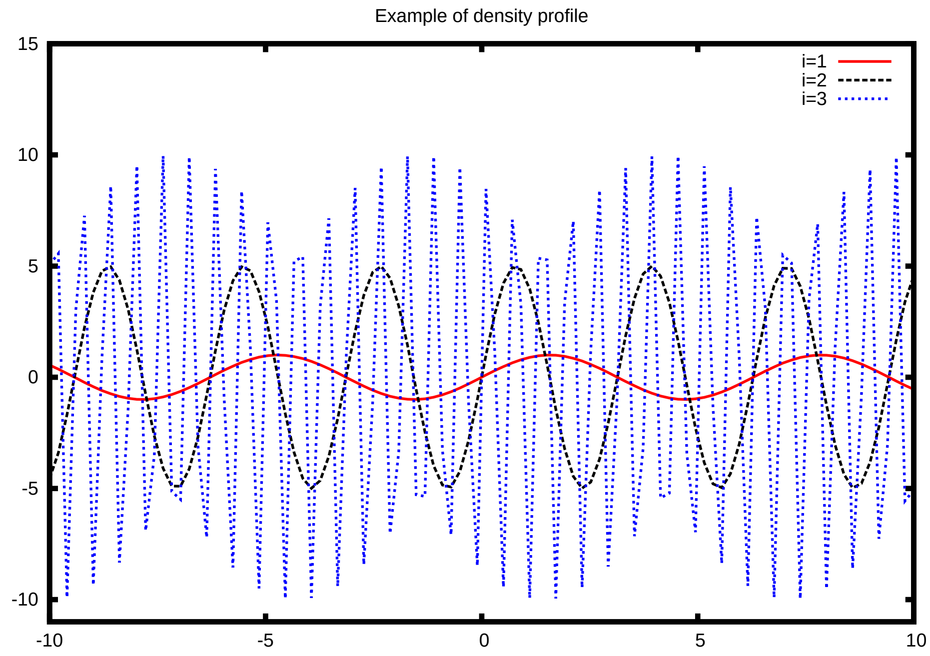
- Example of $h_{ab}(\lambda, x)$ behaviour: $\lambda \sin(x/\lambda)$

Problems with interpretation

- Example of $h_{ab}(\lambda, x)$ behaviour: $\lambda \sin(x/\lambda)$
- Second derivatives: $(1/\lambda)\sin(x/\lambda)$ – oscillations amplitude $\to \infty$

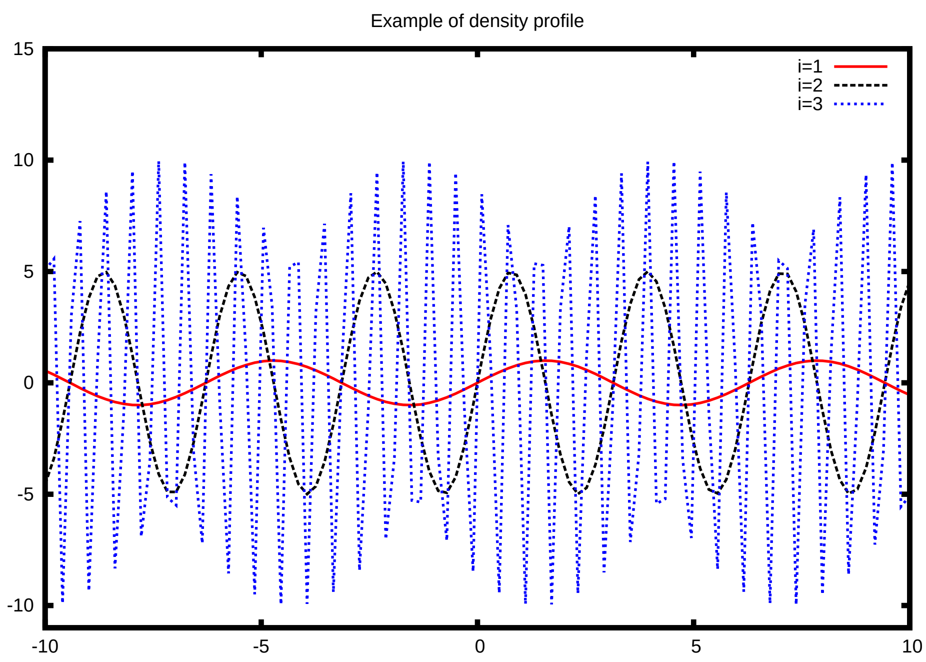
Problems with interpretation

- Example of $h_{ab}(\lambda, x)$ behaviour: $\lambda \sin(x/\lambda)$
- Second derivatives: $(1/\lambda)\sin(x/\lambda)$ – oscillations amplitude $\to \infty$



Problems with interpretation

- Example of $h_{ab}(\lambda, x)$ behaviour: $\lambda \sin(x/\lambda)$
- Second derivatives: $(1/\lambda)\sin(x/\lambda)$ – oscillations amplitude $\to \infty$



- w–lim $T_{ab}(\lambda) = T_{ab}^{(0)}$? - averaging over inhomogeneities that were not originally there

Further reading

For further details see: Is there proof that backreaction of inhomogeneities is irrelevant in cosmology? by T. Buchert et al. (arXiv:1505.07800[gr-qc])

Summary

- RZA provides a potentially powerful tool for describing the large scale structure of the Universe
- Intrinsic curvature plays a role in the evolution of the scale factor
- Small metric perturbations may cause significant curvature deviations and thus deviate from the homogeneous model
- The 'inhomog' code will provide a tool for RZA calculations