



Thesis for the degree of Doctor of Philosophy
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Thermodynamic methods for relativistic hydrodynamics

By

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Declaration

I hereby declare that the contents and organization of this thesis constitute my own original work and do not compromise in any way the rights of third parties, including those relating to the security of personal data.

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I would like to dedicate this thesis to my lovely girlfriend, Aleksandra Kotek

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Abstract

Relativistic hydrodynamics is an essential tool for modelling many astrophysical systems, including accretion disks, neutron stars, and the Universe itself. Hydrodynamic simulations are the only means by which we can extract solid quantitative predictions for high-energy astrophysical phenomena, such as supernovae, jets, and neutron star mergers.

We can consider relativistic hydrodynamics as the point of intersection of thermodynamics and classical field theory. Unfortunately, implementing the irreversible character of the first, in a way that is consistent with the mathematical requirements of the second (e.g. causality and well-posedness of the initial value problem), is a difficult task. Many differential equations, which seem to be physically motivated (such as the heat equation), are not exploitable as relativistic field theories.

This work addresses two major open problems in relativistic hydrodynamics: thermodynamic consistency (i.e. making sure that all thermodynamic principles are obeyed) and stability (i.e. making sure that fluids do not depart spontaneously from thermodynamic equilibrium). We develop several methods and diagnostics for testing whether a theory is physically meaningful, and well suited for numerical implementation. The most important result of the thesis is that, for a large class of theories, stability is mathematically equivalent to thermodynamic consistency.

Abstract

Hydrodynamika relatywistyczna jest niezbędnym narzędziem do modelowania wielu układów astrofizycznych, w tym dysków akrecyjnych, gwiazd neutronowych i samego Wszechświata. Symulacje hydrodynamiczne to jedyny sposób, za pomocą którego możemy uzyskać solidne prognozy ilościowe dotyczące wysokoenergetycznych zjawisk astrofizycznych, takich jak supernowe, dżety i łączenie się gwiazd neutronowych.

Możemy uznać hydrodynamikę relatywistyczną za punkt przecięcia termodynamiki i klasycznej teorii pola. Niestety, zaimplementowanie nieodwracalnego charakteru pierwszego, w sposób zgodny z matematycznymi wymaganiami drugiego (np. przyczynowość i zagadnienie poprawnie postawione dla wartości początkowej) to trudne zadanie. Wiele równań różniczkowych, które wydają się być motywowane fizycznie (takie jak równanie ciepła), nie nadaje się do wykorzystania jako relatywistyczne teorie pola.

Praca ta dotyczy dwóch głównych otwartych problemów hydrodynamiki relatywistycznej: spójności termodynamicznej (tj. upewnienie się, że wszystkie zasady termodynamiczne są przestrzegane) i stabilności (tj. upewnienie się, że płyny nie odbiegają spontanicznie od równowagi termodynamicznej). Opracowujemy kilka metod i diagnostyk do testowania, czy teoria ma sens fizyczny i dobrze nadaje się do implementacji numerycznej. Najważniejszym wynikiem tej pracy jest to, że dla dużej klasy teorii, stabilność jest matematycznie równoważna spójności termodynamicznej.

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Chapter 1

Introduction

1.1 Hydrodynamics: what is it?

Nearly all astrophysical phenomena can be modelled, at large scales, using hydrodynamics. But... what is hydrodynamics?

Let us focus our attention on highly energetic phenomena, and/or on systems which are in the strong-gravity limit. In this case, the key quantity which carries most of the information an astrophysicist is interested in is the (symmetric) stress-energy tensor $T^{\mu\nu}$ [1–3]. The importance of $T^{\mu\nu}$ for astrophysical considerations cannot be overestimated. It is the source of gravity on the right-hand side of the Einstein field equations:

$$G^{\mu\nu} = 8\pi T^{\mu\nu} \quad (G^{\mu\nu} \text{ is the Einstein tensor}). \quad (1.1)$$

Therefore, not only $T^{\mu\nu}$ determines the gravitational field inside all astrophysical objects [4], but it is also a source of gravitational waves [5]. Furthermore, $T^{\mu\nu}$ carries information about the thermodynamic conditions of matter [6]. Hence, its knowledge is crucial for modelling radiation emissions (electromagnetic [7], gravitational [8] or neutrinos [9]).

In an ideal world, the astrophysicist should be able to associate to a given astronomical object a quantum state $\hat{\rho}$ (i.e. a density operator [10, 11], in the Heisenberg picture), carrying information about all the statistical properties of the system, including all the correlations of its constituent particles. Then, taken the quantum stress-energy tensor operator $\hat{T}^{\mu\nu}$ (computed from the microscopic field

theory's action using Belinfante's prescription [12]), the astrophysics should extract the observable stress-energy tensor $T^{\mu\nu}$ as the average [13]

$$T^{\mu\nu} = \text{Tr}[\hat{\rho} \hat{T}^{\mu\nu}]. \quad (1.2)$$

Unfortunately, astronomical objects contain a huge number of particles N (e.g. the Sun contains $N \sim 10^{57}$ hydrogen atoms), so that the explicit computation of any non-equilibrium density operator $\hat{\rho}$ is out of the question.

Luckily, the limit $N \rightarrow +\infty$ enables the emergence of a new effective theory: hydrodynamics. The principle at the heart of hydrodynamics is simple: due to thermodynamic relaxation, most degrees of freedom of the system relax to their equilibrium value over *microscopic* time-scales (namely, within the collision time τ_{coll} [14]). Hence, if the process under consideration is sufficiently slow ($\tau_{\text{process}} \gg \tau_{\text{coll}}$), the state of the system can be completely characterised by the values of those very few degrees of freedom φ_i that relax over *macroscopic* time-scales (namely, time-scales that are much larger than τ_{coll} [15]). Such “slow” degrees of freedom φ_i play the role of effective fields, and the low-frequency field theory for such fields is what we call “hydrodynamics” [16].

The procedure for constructing a hydrodynamic description of a fluid consists of three steps:

Step 1 - Find the relevant effective fields φ_i of the system. There is one effective field φ_i for each conservation law (see figure 1.1), e.g.

$$\begin{aligned} \text{Conservation of Energy} &\longrightarrow T && \text{(temperature)} \\ \text{Conservation of Momentum} &\longrightarrow u^\mu && \text{(flow velocity)} \\ \text{Conservation of Baryon Number} &\longrightarrow \mu_b && \text{(baryon chemical potential)} \\ \text{Conservation of Electric Charge} &\longrightarrow \mu_q && \text{(charge chemical potential)} \\ \text{Superfluid Winding Number} &\longrightarrow w^\mu && \text{(super-flow)}. \end{aligned} \quad (1.3)$$

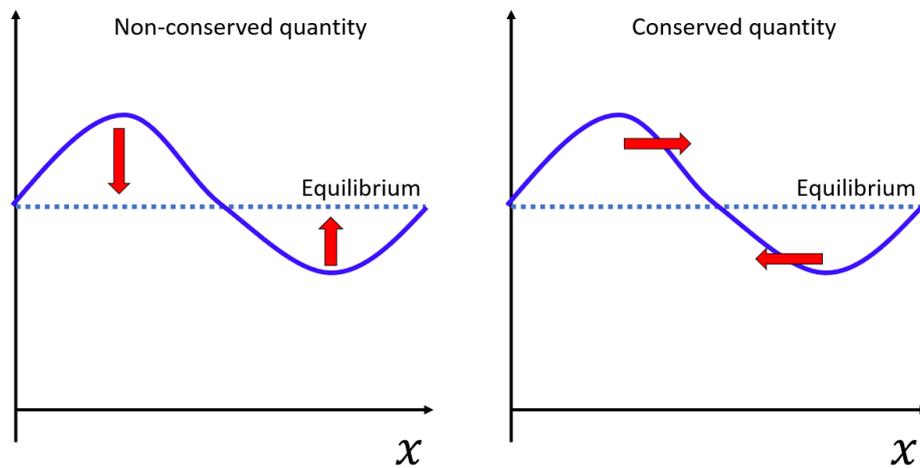


Fig. 1.1 Left panel: If a non-conserved quantity is out of equilibrium, it can just relax to equilibrium locally, through collisions; in such relaxation process, there is no need for different regions of the system to communicate with each other, and the relaxation time-scale will be τ_{coll} . Right panel: If, on the other hand, a conserved quantity is out of equilibrium, the only way for it to equilibrate is that the regions with an excess of such quantity transfer their excess to the regions with a deficit. However, if the system is large, it can take a lot of time for the conserved quantity to travel across it. This implies that, in the limit of large systems and small gradients, the relaxation time-scale of a conserved quantity is much longer than τ_{coll} [17].

Furthermore, there is an additional effective field φ_i for each quasi-conservation law¹ [18], e.g.

$$\begin{aligned} \text{Chemical Reaction} &\longrightarrow \mathbb{A} && \text{(reaction affinity)} \\ \text{Friction} &\longrightarrow u_X^\mu && \text{(additional flows)} \\ \text{Crystal structure} &\longrightarrow X^i && \text{(matter space)}. \end{aligned} \quad (1.4)$$

Finally, if the system is crossed by long-lasting macroscopic electric currents, one should also include the electromagnetic four-potential A_μ among the fields φ_i .

Step 2 - Since we are using the fields φ_i to characterise the state of the system, we also need to write $T^{\mu\nu}$ as a function of the state:

$$T^{\mu\nu} = T^{\mu\nu}(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \dots). \quad (1.5)$$

This formula is called *constitutive relation* for $T^{\mu\nu}$, and it connects the hydrodynamic model (which is a purely mathematical construction) with “real physics”. In fact, equation (1.5) effectively replaces equation (1.2), and allows the astrophysicist to extract observables directly from the hydrodynamic analysis. Note that, if the system has some (quasi-)conserved currents J_X^μ , and an entropy current s^μ , one may also be interested in the additional constitutive relations

$$J_X^\mu = J_X^\mu(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \dots) \quad s^\mu = s^\mu(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \dots). \quad (1.6)$$

Equations (1.5) and (1.6) need to be computed from statistical mechanics and kinetic theory.

Step 3 - The final ingredient that we need is a prescription for how the system evolves in time. Such prescription should take form of a set of equations

$$\mathcal{F}_h(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \dots) = 0, \quad (1.7)$$

¹Quasi-conservation law: a situation in which a quantity is not exactly conserved, but the microscopic processes which break its conservation are rare, so that the relaxation process occurs over hydrodynamic time-scales.

called *field equations* (or, sometimes, hydrodynamic equations). Each conservation law provides one field equation. For example, the energy-momentum conservation produces four field equations:

$$\nabla_{\mu} T^{\mu\nu} = 0. \quad (1.8)$$

However, if there are some quasi-conservation laws, one needs a kinetic model for how the quasi-conserved quantity is dissipated, and this will lead to more complicated field equations. For example, if the conservation of a current J^{μ} is broken by the existence of a chemical reaction (with reaction affinity \mathbb{A}), there will be a field equation of the form [19]

$$\nabla_{\mu} J^{\mu} - \Xi \mathbb{A} = 0, \quad (1.9)$$

where the coefficient Ξ needs to be computed through chemical kinetics.

Equations (1.5), (1.6) and (1.7) define what we call a “hydrodynamic theory”.

1.2 Relativistic hydrodynamics and astrophysics

Now that we have an idea of what hydrodynamics is, let us have a brief overview of the astrophysical phenomena that require relativistic hydrodynamic modelling. We will focus on those research areas that can be influenced the most by an advancement on the mathematical and foundational sides.

1.2.1 Core-collapse supernovae

Basic properties:

Typical temperatures: $T \sim 50 \text{ MeV} = 5 \times 10^{11} \text{ K}$

Typical densities: $\rho \sim 10^{14} \text{ g/cm}^3$ [for a neutron star remnant], $\rho \rightarrow +\infty$ [for a black hole remnant]

Lifespan: $\Delta t \sim 100 \text{ ms}$ [collapse] + 10 s [cooling of proto-neutron star]

Messengers: Electromagnetic radiation, neutrinos, gravitational waves

Most relevant fields: $\varphi_i = \{T, u^\mu, \mu_p, \mu_n, \mu_e\}$ + neutrino kinetic theory

A self-gravitating fluid can sustain itself against the gravitational pull only by means of pressure gradients. In fact, for spherically-symmetric systems in hydrostatic equilibrium, the Tolman-Oppenheimer-Volkoff (TOV) equation for the pressure reads [6]

$$\frac{dp}{dr} = -(\rho + p) \frac{d\phi}{dr} \quad [p = \text{pressure}, \rho = \text{energy density}, r = \text{radius}] \quad (1.10)$$

where $\phi := \log(-g_{tt})/2$ is the gravitational potential. Equation (1.10) is a direct consequence of (1.8). On the other hand, for simple fluids one has $p = p(T, n)$ [n = particle density], so that

$$\frac{dp}{dr} = \left. \frac{\partial p}{\partial T} \right|_n \frac{dT}{dr} + \left. \frac{\partial p}{\partial n} \right|_T \frac{dn}{dr}. \quad (1.11)$$

This means that both temperature and density gradients can contribute to sustain a fluid.

In a star, the temperature gradients are large, as a result of the heat generated by the nuclear reactions, and released through the star's surface (recall the Fourier law: $\nabla T \propto -q$, where q is the heat flux). However, when these nuclear reactions reach chemical equilibrium, nothing can prevent the star's core from evolving towards thermal equilibrium, defined by the condition [20]

$$Te^\phi = \text{const} \quad [\text{in a star, } \phi \text{ is close to } 1]. \quad (1.12)$$

In this new equilibrium configuration, there are essentially no temperature contributions to dp/dr [see equation (1.11)], and most of the pressure in the core is provided by the degenerate electron gas. When the compacted core exceeds the Chandrasekhar limit (roughly $1.4 M_\odot$), the degenerate electrons become ultra-relativistic. Such a configuration is unstable, and collapses rapidly, leading to a violent explosion. This process is called “core-collapse supernova” and occurs only in stars with mass $8M_\odot \lesssim M \lesssim 40M_\odot$.

What happens in the core of a star undergoing supernova explosion can be understood only through hydrodynamic simulations [21]. Both special and general relativity must be accounted for, since we are both in the high-energy and in the strong-gravity limit (at least close to center of the star). It is also fundamental to track

the chemical composition of matter explicitly, because neutrinos play a fundamental role in the process, and these interact with other types of matter through chemical relations, like



One needs to track also the entropic content of the fluid elements, because the temperature grows rapidly as the core contracts. The whole hydrodynamic process is so complicated that it is even hard to predict whether the hydrodynamic simulation will manage to get the supernova to actually explode [22]!

The ultimate result of a core-collapse supernova is a newly-born neutron star or a newly-born black hole. In the case of a neutron star, it is important also to simulate its first 20 seconds of life [23], because during this time the neutron star cools down very rapidly (emitting most of the neutrinos that are, then, detected), and the end state of this cooling process is the initial state of the secular thermal evolution of the neutron star.

1.2.2 Cold neutron stars

Basic properties:

Typical temperatures: $T \sim 10^8$ K

Typical densities: $\rho \sim 10^{14}$ g/cm³

Lifespan: Possibly the age of the Universe

Messengers: Electromagnetic radiation, gravitational waves (continuous waves are yet to be observed)

Most relevant fields: $\varphi_i = \{u^\mu, \mu_p, \mu_n, w^\mu, X^i, A_\mu\}$

Neutron stars are one possible remnant of a type-II supernova. They have masses that range between one and two M_\odot and radii of about 10 km [24]. Their relativistic compactness is

$$\frac{GM}{Rc^2} \sim 0.2, \quad (1.14)$$

which implies that general-relativistic corrections are relevant. The gravitational time-dilation factor in the center of a neutron star is

$$\frac{[\Delta\tau]_{r=+\infty}}{[\Delta\tau]_{r=0}} \approx \frac{3}{2}. \quad (1.15)$$

This tells us that general-relativistic corrections to some observables can be of the order of 50% [25]. Furthermore, even if we work at scales where the spacetime curvature is negligible, close to the center of a neutron star the pressure is of the same order of the energy density. Hence, special-relativistic effects cannot be neglected. For example, if we want to model the propagation of sound-waves inside a neutron star's core, we cannot use Newtonian hydrodynamics, because this would lead to a wrong formula for the speed of sound [6] (m_b is the baryon mass):

$$(c_s^2)_{\text{Relativity}} = \left. \frac{\partial p}{\partial \rho} \right|_{s/n} = \frac{m_b n}{\rho + p} \left. \frac{\partial p}{\partial (m_b n)} \right|_{s/n} = \frac{m_b n}{\rho + p} (c_s^2)_{\text{Newton}} \sim \frac{1}{2} (c_s^2)_{\text{Newton}}. \quad (1.16)$$

The typical density of a neutron star is of the order of the nuclear density ($\rho_N \approx 2 \times 10^{14} \text{ g/cm}^3$). At these densities, the separation of matter into individual nuclei is no longer possible [24], and one has a single aggregate of baryons and leptons, called “nuclear matter”. Compared with earthly temperatures, the typical temperature of a neutron star is high: $T \sim 10^8 \text{ K}$. However, due to the high densities, such temperature is much below the neutron's Fermi temperature ($\sim 10^{12} \text{ K}$). Hence, neutron star matter is strongly degenerate, and one can adopt the zero-temperature approximation.

These thermodynamic conditions allow for the presence of several phases of matter [26]: solid phase (outermost layers), superfluid-solid mixture (outer layers), and superfluid-superconductive mixture (inner layers). Hence, modelling any astrophysical process involving a cold neutron star (like starquakes, neutron star oscillations and pulsar glitches) requires the use of complicated hydrodynamic theories, with many degrees of freedom, accounting for the effect of all such phases. The hydrodynamic formalism which can be adopted to model most of these additional degrees of freedom is Carter's multifluid approach [27–31], which has been designed with the purpose of extending the perfect-fluid theory to fluids having an arbitrary number of non-collinear currents [19, 32].

1.2.3 Neutron-star mergers

Basic properties:

Typical temperatures: $T \sim 10 \text{ MeV} = 10^{11} \text{ K}$ (but they may reach 100 MeV)

Typical densities: $\rho > 10^{14} \text{ g/cm}^3$

Lifespan: $\Delta t \sim 10 \text{ s}$ [observable chirp] + 100 ms [actual merger] + 100 s for a hypermassive neutron star or $+\infty$ for stable NS [post-merger object]

Messengers: Electromagnetic radiation, neutrinos, gravitational waves

Most relevant fields: $\varphi_i = \{T, u^\mu, \mu_p, \mu_n, \mu_e, A_\mu\}$

Consider a binary system of compact objects (like neutron stars and black holes). Let us work under the approximation that the two objects have the same mass. Then, their orbital frequency around the center of mass of the system is

$$\Omega = \sqrt{\frac{M}{4R^3}} \quad [R = \text{distance from the center of mass}]. \quad (1.17)$$

As the two objects move, the total quadrupole moment of the system changes with time, with period² $\tau = \pi/\Omega$. Hence, we can roughly estimate the time-dependent part of the gravitational field that they generate [3], in the far-field approximation,

$$\bar{h}^{jk} \sim \frac{M^2}{Rr} \begin{bmatrix} -2 \cos(2\Omega t_R) & -2 \sin(2\Omega t_R) & 0 \\ -2 \sin(2\Omega t_R) & 2 \cos(2\Omega t_R) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (1.18)$$

Here, r is the distance from the binary system, $t_R = t - r$ is the retarded time, and \bar{h} is the trace-reversed metric perturbation. The coordinate system has been chosen in a way that the x^3 -axis is also the symmetry axis of the binary system.

In asymptotically-flat spacetimes, one can associate a total four-momentum to an isolated body (in our case the body is the binary system as a whole), by means of the

²It is not $\tau = 2\pi/\Omega$ because the objects have the same mass, thus it takes only half of a revolution for the quadrupole moment to come back to the initial value.

formula [1]

$$P^\mu = \frac{1}{16\pi} \int \partial_\alpha \partial_\beta H^{\mu\alpha 0\beta} d^3x \quad (1.19)$$

with $H^{\mu\alpha\nu\beta} = -(\bar{h}^{\mu\nu} \eta^{\alpha\beta} + \eta^{\mu\nu} \bar{h}^{\alpha\beta} - \bar{h}^{\alpha\nu} \eta^{\mu\beta} - \bar{h}^{\mu\beta} \eta^{\alpha\nu})$.

Since $H^{\mu\alpha\nu\beta}$ is antisymmetric in ν and β , one has that $\partial_\alpha \partial_\beta \partial_\nu H^{\mu\alpha\nu\beta} = 0$, so that the four-momentum obeys the conservation law

$$\frac{dP^\mu}{dt} = -\frac{1}{16\pi} \oint_{\mathcal{S}} \partial_\alpha \partial_\beta H^{\mu\alpha j\beta} d^2\mathcal{S}_j, \quad (1.20)$$

where \mathcal{S} is the boundary of a large region of space enclosing the body. Evaluating this integral in rest-frame of the source (one needs to go beyond the linear approximation, to obtain a result that is not zero), one finds that gravitational waves carry away energy [1]: $dP^0/dt < 0$. As a result, the orbit of the two compact objects shrinks. However, we see from (1.18) that, the smaller the orbit's radius R , the larger the gravitational-wave emission, which causes the orbit to shrink faster and faster. At one point, the two objects touch each other. This is what we call a “merger”.

If at least one of the two objects is a neutron star, the merger is a hydrodynamic process, which needs to be simulated numerically. The “hydrodynamic nature” of the process leaves important imprints in the observed gravitational wave-form, which allow one to impose observational constraints on the (still mysterious) equation of state of neutron-star matter [33, 34].

In the few instants before the merger, tidal forces are very strong; they deform the star in a time-dependent way, inducing oscillations that heat up the matter. This heating melts the solid structure of the outer layers and destroys the superfluid phases, allowing one to adopt more standard hydrodynamic theories than those that are needed to model cold neutron stars [35]. However, during the merger, fluid elements travel rapidly between regions at high pressure and regions at low pressure, so that they expand and contract faster than the time necessary for the chemical fractions to reach chemical equilibrium for the given pressure. Hence, it is necessary, in a merger, to keep track explicitly of all the chemical components (mainly protons neutrons and electrons), since the chemical evolution leads to dissipation [36].

Usually, the post-merger remnant is a black hole. However, there is a possibility that, if the merging objects are two very light neutron stars (with mass close to $1 M_\odot$),

the post-merger remnant might be a highly-massive neutron star [37]. Such neutron star may be below the maximum mass ($\sim 2.3 M_{\odot}$) allowed by the TOV equations (in which case it may survive indefinitely), or above it. In the second case, we talk about a “super-massive” neutron star: a neutron star which resists the gravitational collapse because of its fast rotation. An object of this kind is usually metastable and short-lived [38], because copious neutrino/gravitational/electromagnetic emission carry away angular momentum, slowing down the rotation.

1.2.4 Universe

Basic properties:

Typical temperatures: $T \sim 10^{32}$ K [at $\tau \sim 0$], 10^{10} K [at $\tau = 1$ s], 3 K [CMB now]

Typical densities: $\rho \sim \infty$ [at $\tau \sim 0$], 10^{-6} g/cm³ [at $\tau \sim 10$ s], 10^{-29} g/cm³ [now]

Lifespan: $\Delta\tau \sim 10^{10}$ yrs [till now]

Messengers: Electromagnetic radiation, (maybe) gravitational waves

Most relevant fields: $\varphi_i = \{T, \rho_b, \rho_{DM}, \Lambda, \Pi\}$ + inflaton degrees of freedom

The expansion of the Universe can be successfully modelled only within General Relativity. In the simplest model possible, one assumes that the large-scale structure of the Universe is homogeneous and isotropic, so that one can adopt the Friedmann-Lemaitre-Robertson-Walker metric:

$$ds^2 = -d\tau^2 + a^2(\tau) \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (1.21)$$

where k is the curvature constant of space. For such metric, the Einstein field equations (1.1) become the Friedman equations

$$\begin{aligned} \frac{3\dot{a}^2}{a^2} &= 8\pi\rho - \frac{3k}{a^2} \\ \frac{3\ddot{a}}{a} &= -4\pi(\rho + 3p). \end{aligned} \quad (1.22)$$

As we can see, both the total energy density ρ and the total pressure p of all types of matter contribute to determine the evolution of the expansion rate of the Universe $H = \dot{a}/a$ (Hubble parameter). Therefore, one needs to model carefully both baryon contributions (ρ_b, T) , dark matter contributions (ρ_{DM}) , dark energy contribution $(\rho_{DE} = \Lambda/8\pi)$ and any additional field degree of freedom which could be relevant in the early Universe, like the inflaton. However, it is also important to realise that the Universe is an out-of-equilibrium system. Hence, in general one may observe deviations from thermodynamic equilibrium affecting the value of the total pressure p (especially in the early Universe, when approximating baryonic matter to a dust is not an option). Such non-equilibrium corrections to p are called “bulk viscosity”, and are denoted by the symbol Π , whose formal definition is [39, 40]

$$\Pi := p_b - p_b^{\text{eq}}(\rho_b, n_b), \quad (1.23)$$

where $p_b^{\text{eq}}(\rho_b, n_b)$ is the equilibrium equation of state, which determines the equilibrium pressure for given energy density and baryon density. It is mostly this effect, namely the presence of a non-vanishing Π , that connects cosmology with the theory of relativistic viscous hydrodynamics [41–43].

1.3 Challenges of relativistic hydrodynamics

A very counter-intuitive aspect of relativistic hydrodynamics is that many theories, which look natural at the first glance, turn out to be pathological and unreliable. There is a pitfall at every corner! In many specific cases, the mathematical origin of the pathology has been singled out, but a systematic discussion of the most important sources of problems is still missing. This is the gap that my thesis aims to fill. In particular, this thesis has the goal of explaining the origin of the most widespread pathologies in relativistic hydrodynamic theories. Furthermore, it aims at providing quick diagnostics for making sure that a given hydrodynamic theory “makes sense” and can actually be employed in astrophysical models and simulations.

Let us begin by reviewing the three main sources of problems that one can meet while building a theory.

1.3.1 Thermodynamic consistency

The most common relativistic hydrodynamic theory one can come across (at least in astrophysical problems) is the perfect-fluid theory at finite chemical potential. Let us briefly summarise how it is constructed, following the three-step procedure outlined in section 1.1.

The theory describes the evolution of a fluid with one conserved scalar charge (Σ : space-like Cauchy surface)

$$Q = \int_{\Sigma} J^{\mu} d\Sigma_{\mu} \quad (\text{orientation: } d\Sigma_0 > 0). \quad (1.24)$$

The conservation law for Q leads to the introduction of a corresponding field degree of freedom μ : the chemical potential. Besides the conservation of Q , one also assumes conservation of energy and momentum, so that two other fields are needed³: T and u^{μ} . No other conservation law is assumed, nor any quasi-conservation law. Large-scale electromagnetic effects are also not considered, so that μ , T and u^{μ} completely characterise the state of the system. This completes step 1. But it also completes step 3, because we have as many conservation laws as fields: no additional field equation is required, besides $\nabla_{\mu} J^{\mu} = \nabla_{\mu} T^{\mu\nu} = 0$.

Step 2 is carried out invoking two crucial assumptions:

- (i) The local properties of a perfect fluid at a point x depend only on the local thermodynamic conditions $\varphi_i(x)$;
- (ii) In the homogeneous limit, the hydrodynamic theory should be equivalent to the description that one finds in standard thermodynamics textbooks [44] (correspondence criteria: $U \equiv VT^{00}$, $N \equiv VJ^0$, $S \equiv Vs^0$).

In essence, condition (i) is telling us that the constitutive relations of a perfect fluid are “ultra-local” [45]: they do not involve the gradients of φ_i , but only their local values. This implies, for example, that $J^{\mu} \propto s^{\mu} \propto u^{\mu}$, because u^{μ} is the only vector field of the theory.

³One can think of energy and momentum as orthogonal projections of a four-dimensional conservation law: the four-momentum. In this case, one may naturally combine the two effective fields T and u^{μ} into the inverse-temperature four-vector $\beta^{\mu} = u^{\mu}/T$ [11].

It is straightforward to verify that there is only one possible theory that obeys conditions (i,ii):

$$\begin{aligned}
\varphi_i &= \{T, u^\mu, \mu\} \\
p &= p(T, \mu) \text{ with } dp = s dT + n d\mu \\
T^{\mu\nu} &= (sT + n\mu)u^\mu u^\nu + pg^{\mu\nu} \\
J^\mu &= nu^\mu \\
s^\mu &= su^\mu.
\end{aligned} \tag{1.25}$$

The explicit formula $p = p(T, \mu)$ changes from fluid to fluid, and is called ‘‘equation of state’’.

The consistency of the perfect-fluid theory with thermodynamic principles was enforced ab-initio, by means of condition (ii). But, what would happen if we ignored condition (ii), and just decided to build the most general field theory compatible with (i)? In this case, we would still end up with constitutive relations of the form

$$\begin{aligned}
T^{\mu\nu} &= (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \\
J^\mu &= nu^\mu \\
s^\mu &= su^\mu.
\end{aligned} \tag{1.26}$$

However, ρ , p , n , and s now would be *unrelated* functions of T and μ . It turns out that a generic theory of this kind violates the second law of thermodynamics. To see this, we can perform the change of variables

$$(T, u^\mu, \mu) \longrightarrow (s, u^\mu, n), \tag{1.27}$$

and insert the conservation laws (notation: $\dot{A} = u^\mu \nabla_\mu A$, $d\rho = \rho_s ds + \rho_n dn$)

$$\begin{aligned}
\nabla_\mu J^\mu = 0 &\quad \implies \quad \dot{n} = -n \nabla_\mu u^\mu \\
u_\nu \nabla_\mu T^{\mu\nu} = 0 &\quad \implies \quad \dot{\rho} = \rho_s \dot{s} + \rho_n \dot{n} = -(\rho + p) \nabla_\mu u^\mu
\end{aligned} \tag{1.28}$$

into $\nabla_\mu s^\mu$, to obtain

$$\nabla_\mu s^\mu = s \nabla_\mu u^\mu + \dot{s} = \frac{\rho_s s + \rho_n n - \rho - p}{\rho_s} \nabla_\mu u^\mu. \tag{1.29}$$

We see that the entropy production rate can have arbitrary sign ($\nabla_\mu u^\mu$ may be positive or negative), unless we require that

$$\rho + p = \rho_s s + \rho_n n. \quad (1.30)$$

This condition is obviously respected by the perfect-fluid theory (1.25).

The example above shows that not every possible constitutive relation is physically meaningful, and one needs to pay particular attention that the final theory will be consistent with thermodynamic principles. However, the compatibility with the second law of thermodynamics ($\nabla_\mu s^\mu \geq 0$) is only one aspect of the complicated subject called “thermodynamic consistency”. In fact, thermodynamics provides a number of inequalities that need to be respected by the hydrodynamic theory, otherwise the system may exhibit paradoxical behaviours. Let us consider one more example.

Suppose that we make a simulation of neutron-star oscillations, modelling hadronic matter as a perfect fluid. Let us imagine that we are adopting a wrong equation of state, such that the specific heat at constant pressure is negative:

$$c_p = T \left. \frac{d(s/n)}{dT} \right|_p < 0. \quad (1.31)$$

This is an example of a system that is hydrodynamically stable but thermodynamically unstable. The fact that it is hydrodynamically stable means that the simulation (if correctly designed) will not show any pathological behaviour: the star will oscillate adiabatically according to the expectations. However, if we decide to refine our simulation, and implement also neutrino transport, big problems will surely appear. In fact, hotter regions of the star produce and emit more neutrinos (the production rates of neutrinos grow with the temperature [46]); as neutrinos are emitted, the hotter regions lose energy; but the heat capacity is negative, so lower energy implies higher temperature, which in turn leads to a faster emission of neutrinos from the hotter regions, leading to a thermal runaway [40].

Thermodynamic consistency automatically rules out paradoxical behaviours of this kind. Indeed, $c_p \geq 0$ is a well-known thermodynamic inequality, which should be obeyed by every fluid [15]. Such inequality has also a statistical origin: in the Isothermal-Isobaric (NpT) ensemble, the heat capacity at constant pressure is a

measure of the squared uncertainty of the system's entropy [47],

$$Nc_p = \overline{(\delta S)^2} \geq 0, \quad (1.32)$$

and must, therefore, be non-negative.

For simple perfect fluids, a list of inequalities of this kind (like $c_p \geq 0$) is provided in any thermodynamics text-book [one just needs to invoke condition (ii)]. But, what happens if we add to the theory a new, exotic, degree of freedom? For example, what do the thermodynamic inequalities look like, when we add the shear stresses $\Pi_{\mu\nu}$ to the set of independent fields φ_i ? Or, if we allow for the formation of relative flows, what are the additional thermodynamic inequalities that the system needs to respect?

This is the central problem at the heart of this thesis, and will be formally solved in Chapter 7.

1.3.2 Stability

Let us work, for simplicity, in 1+1 dimensions, and consider the heat equation

$$\partial_t T = D \partial_x^2 T. \quad (1.33)$$

Let us see how this equation behaves when we boost it, i.e. we look at the evolution of temperature profiles in a reference frame in which the conductive material is moving with velocity $-v$.

The Lorentz boost in 1+1 dimensions reads

$$\begin{aligned} \tilde{t} &= \gamma(t - vx) & \longleftrightarrow & & t &= \gamma(\tilde{t} + v\tilde{x}) \\ \tilde{x} &= \gamma(x - vt) & & & x &= \gamma(\tilde{x} + v\tilde{t}), \end{aligned} \quad (1.34)$$

so that the partial derivatives transform as follows:

$$\partial_t = \gamma \partial_{\tilde{t}} - \gamma v \partial_{\tilde{x}} \quad \partial_x = \gamma \partial_{\tilde{x}} - \gamma v \partial_{\tilde{t}}. \quad (1.35)$$

Interpreting T as a scalar field, equation (1.33) becomes

$$\partial_{\tilde{t}} T - v \partial_{\tilde{x}} T = \gamma D (\partial_{\tilde{x}}^2 T - 2v \partial_{\tilde{x}} \partial_{\tilde{t}} T + v^2 \partial_{\tilde{t}}^2 T). \quad (1.36)$$

We note that, in the new reference frame, the heat equation is of second order in time. This means that, when we specify the initial conditions of the system, the function $\partial_{\tilde{t}}T(\tilde{t}=0)$ is no longer determined by fixing the function $T(\tilde{t}=0)$. Instead, we need to fix both $T(\tilde{t}=0)$ and $\partial_{\tilde{t}}T(\tilde{t}=0)$, independently. In other words, the state-space of the system has doubled dimension with respect to the state-space of the Newtonian boosted theory⁴. This leads to the insurgence of new solutions that have no Newtonian analogue. To see this, let us (for simplicity) restrict our attention to solutions of the form

$$T = T(\tilde{t}). \quad (1.37)$$

Then, equation (1.36) becomes $\partial_{\tilde{t}}T = \gamma Dv^2 \partial_{\tilde{t}}^2 T$, whose general solution is a linear combination of two fundamental solutions:

$$\begin{aligned} T = \text{const} & \quad \text{“Newtonian solution”} \\ T = \exp\left(\frac{\tilde{t}}{D\gamma v^2}\right) & \quad \text{“Non-Newtonian solution”}. \end{aligned} \quad (1.38)$$

The first solution is the behaviour that one would expect: a system with uniform temperature (whether it is moving or not) does not conduce heat, and T remains constant. The second solution, however, is unexpected: the temperature of the system grows exponentially, without any bound, and without there being any spatial gradient. The apparent paradox of a heat conduction without gradients is resolved if we consider the relativity of simultaneity [48]: a system that is homogeneous but non-stationary in one reference frame, is not homogeneous in another frame, which means that in the rest frame of the material *there are* spatial gradients, and the Fourier law $\nabla T \propto -q$ holds only in the material’s rest frame. Indeed, if we boost the non-Newtonian solution back to the rest frame of the material, we obtain

$$T = \exp\left(\frac{t - vx}{Dv^2}\right), \quad (1.39)$$

which is a perfectly legitimate non-homogeneous solution of equation (1.33).

The problem is that this solution is completely unphysical. To see why, consider the limit $v \rightarrow 0$, in which we should recover the purely Newtonian description. Then,

$$T = \exp\left[-\frac{1}{Dv}(x - t/v)\right] \longrightarrow \exp[-\infty \times (x - \infty t)]. \quad (1.40)$$

⁴Galilei-boosted heat equation: $\partial_{\tilde{t}}T - v \partial_{\tilde{x}}T = D \partial_{\tilde{t}}^2 T$. Clearly, it is still of first order in time.

This describes an infinite step [$\exp(\infty \times z) \sim \infty \times \Theta(z)$, Θ is the Heaviside step-function] that is travelling at infinite speed, which is clearly unphysical.

Now we immediately see the problem: in the material's rest frame, solutions like (1.39) are automatically ruled out if we require that the initial state of the system has a well-defined Fourier transform (namely, it does not have infinite tails for large $|x|$); on the other hand, in the boosted frame there is no simple way of removing the unphysical part of the solution. A general initial condition will contain also a non-Newtonian part, and we would need to fine-tune the initial data to get rid of unphysical contributions [namely, by setting $\partial_{\tilde{t}}T(\tilde{t} = 0) \equiv 0$]. It is easy to verify that this same problem occurs for every Fourier mode [49]: for any k , solutions of the form

$$T = e^{i(k\tilde{x} - \omega\tilde{t})} \quad (1.41)$$

exist for two possible choices of ω , one physical and one unphysical. Removing the unphysical part requires that the functions $T(\tilde{t} = 0)$ and $\partial_{\tilde{t}}T(\tilde{t} = 0)$ are related in such a way that all the unphysical Fourier contributions are *exactly* zero. Unfortunately, we cannot rely on approximately physical initial data, because, for small ν , the unphysical solution in (1.38) becomes $T \sim \infty \times \Theta(\tilde{t})$, which means that the unphysical contributions becomes infinitely large in an infinitely small time, spoiling the whole solution immediately (something similar happens for every unphysical Fourier mode [49]).

This is known as the “stability problem” of dissipative hydrodynamics, and it is not a peculiar feature of the heat equation: it is a distinctive feature of a large number of theories [50–53]. It makes such theories essentially useless, because they cannot be solved with approximate methods and they do not have a well-posed initial value problem. Explaining the origin of this instability, and how to eliminate it, is one of the goals of the thesis (see Chapter 6 and 7).

1.3.3 Causality

Consider again the heat equation, namely equation (1.33). Its Green function $G = G(t, x)$, i.e. the solution of the Cauchy problem

$$\begin{cases} \partial_t G = D \partial_x^2 G \\ G(0, x) = \delta(x), \end{cases} \quad (1.42)$$

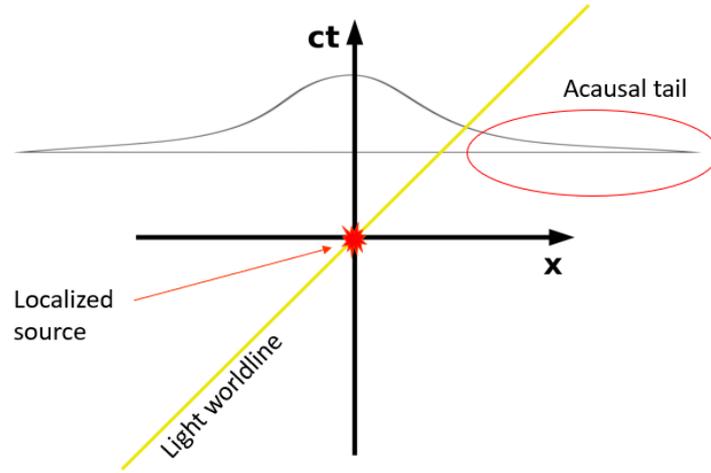


Fig. 1.2 Acausality of the heat equation: An initial Dirac delta of temperature injected in the origin of the coordinates produces, at later times, a Gaussian temperature profile. The tails of this Gaussian exit the light-cone, violating causality.

is given by

$$G(t, x) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \quad (1.43)$$

For any time $t > 0$, this function has infinite support, across all space, although the initial data at $t = 0$ is localised in the origin [$G(0, x) = \delta(x) = 0$ for $x \neq 0$]. This implies that, if we interpret the heat equation “literally”, heat propagates faster than light, see figure 1.2.

In practice, the tails that violate causality are small. Hence, if we keep in mind that the heat equation is just an approximation, these tails should not be a strong reason of concern [54–57]. However, if one is planning to solve a system of partial differential equations for arbitrary initial data, causality violations can be a problem. To understand why, consider the following acausal equation:

$$v^\nu \nabla_\nu \varphi = 0, \quad (1.44)$$

where v^ν is a (given) space-like vector field ($v^\nu v_\nu > 0$) and φ is the scalar field of which we want to determine the evolution. The solution of (1.44) is obviously

$$\varphi = \text{constant along the integral curves of } v^\nu \text{ (which are space-like)}. \quad (1.45)$$

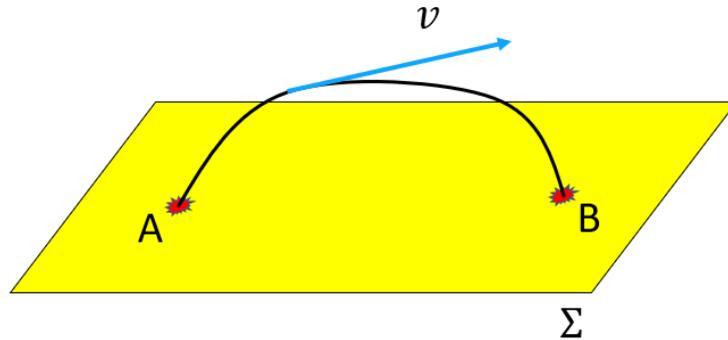


Fig. 1.3 The problem with acausal theories: the initial data on A may propagate to B if the field equation is acausal.

In general, one is interested in studying how an astrophysical system evolves, for a given initial condition. In General Relativity, such initial condition is imposed on a space-like Cauchy 3D-surface Σ . Now, the problem is that, if an integral curve of v^ν intersects Σ on two points A and B (as in figure 1.3), we are not free to impose the initial data on A and B as we like, because we already know, from equation (1.45), that

$$\varphi(A) = \varphi(B). \quad (1.46)$$

Hence, if the principle of causality is violated, the initial conditions on Σ are not arbitrary, and can exhibit unphysical long-range correlations between spatially separated points. This is an important complication, because, to know such correlations, one would need to already know the solution, as in the example above.

This simple argument shows that acausal theories are usually very problematic to handle. It is good practice, in a relativistic setting, to abandon any theory that is not causal [58–61] (even if the causality violations may be, in practice, small), because knowing the unphysical correlations in the initial data is at least as complicated as solving the equation itself. However, many equations in physics (including, for example, the Navier-Stokes equation) are not causal, and obtaining a causal substitute to many of these acausal equations is hard, and involves enlarging the number of degrees of freedom of the theory [62, 39, 63]. Although we will not tackle the issue of causality directly, we will meet it again.

1.4 Outline of the thesis

The thesis is a collection of six papers, published on peer-reviewed journals. Each paper is a chapter:

- Chap. 2 - **L. Gavassino**, Foundations of Physics 50 (2020), *The zeroth law of thermodynamics in special relativity*, doi:10.1007/s10701-020-00393-x
- Chap. 3 - **L. Gavassino**, Foundations of Physics 52 (2021), *Proving the Lorentz invariance of the entropy and the covariance of thermodynamics*, doi:10.1007/s10701-021-00518-w
- Chap. 4 - **L. Gavassino**, M. Antonelli, CQG 37-2 (2019), *Thermodynamics of uncharged relativistic multifluids*, doi:10.1088/1361-6382/ab5f23
- Chap. 5 - **L. Gavassino**, M. Antonelli, B. Haskell, CQG 38-7 (2021), *Bulk viscosity in relativistic fluids: from thermodynamics to hydrodynamics*, doi:10.1088/1361-6382/abe588
- Chap. 6 - **L. Gavassino**, M. Antonelli, B. Haskell, PRD 102 (2020), *When the entropy has no maximum: A new perspective on the instability of the first-order theories of dissipation*, doi:10.1103/PhysRevD.102.043018
- Chap. 7 - **L. Gavassino**, CQG Letters 38-21LT02 (2021), *Applying the Gibbs stability criterion to relativistic hydrodynamics*, doi:10.1088/1361-6382/ac2b0e

The logical path of the thesis is the following.

In Chapters 2 and 3, we lay down the foundations of relativistic thermodynamics in a rigorous way. This is a crucial first step, as we aim to test the thermodynamic consistency of hydrodynamic theories. The principles of relativistic thermodynamics have been formulated, for the first time, by van Kampen [64], who combined intuitive assumptions with powerful insights. The goal of Chapters 2 and 3 is to show that all such principles can be derived *ad initio* if one assumes that there is at least one inertial reference frame in which the second law of thermodynamics ($\dot{S} \geq 0$) is valid: everything follows naturally from that. This result is extremely useful, because it guarantees that, also in relativity, there is nothing more to thermodynamics than the law of non-decreasing entropy and the conservation laws. This implies that the

maximum entropy principle is unchanged in relativity: *the state of thermodynamic equilibrium maximizes the entropy for fixed values of all the integrals of motion.*

In Chapters 4 and 5, we show that the criterion of thermodynamic consistency can be used as a building block of hydrodynamic theories also beyond the perfect-fluid limit. In particular, in Chapter 4 we use the maximum entropy principle to solve some open problems connected with multifluid hydrodynamics. In Chapter 5, we use thermodynamic considerations to extend the Israel-Stewart theory for bulk viscosity to fluids having an arbitrary number of non-equilibrium degrees of freedom.

In Chapters 6 and 7, we reach the heart of the thesis: we prove that (under certain mathematical conditions) thermodynamic consistency is equivalent to stability. Chapter 6 is devoted to verifying that the stability problem of relativistic hydrodynamics arises naturally whenever the maximum entropy principle is violated. In Chapter 7, we finally prove that a hydrodynamic theory is stable if and only if the maximum entropy principle is respected (provided that $\nabla_\mu s^\mu \geq 0$).

Chapter 2

The zeroth law of thermodynamics in special relativity



The Zeroth Law of Thermodynamics in Special Relativity

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Abstract

We critically revisit the definition of thermal equilibrium, in its operational formulation, provided by standard thermodynamics. We show that it refers to experimental conditions which break the covariance of the theory at a fundamental level and that, therefore, it cannot be applied to the case of moving bodies. We propose an extension of this definition which is manifestly covariant and can be applied to the study of isolated systems in special relativity. The zeroth law of thermodynamics is, then, proven to establish an equivalence relation among bodies which have not only the same temperature, but also the same center of mass four-velocity.

Keywords Thermodynamics · Special relativity

1 Introduction

The modern covariant formulation of the second law of thermodynamics relies on the assumption that it is possible to define an entropy four-current s^ν with non-negative divergence, $\nabla_\nu s^\nu \geq 0$ [19, 20]. The validity of this assumption is strongly supported by relativistic kinetic theory, both classical [9] and quantum [10, 13], and by the hydrodynamics of locally isotropic fluids [15], including perfect fluids [42] and chemically reacting fluids [6]. This approach finds application in every branch of the relativistic hydrodynamics, including the problem of relativistic dissipation [21, 31] and multifluid hydrodynamics [1, 7, 8].

In the absence of dissipation ($\nabla_\nu s^\nu = 0$), it is possible to define the total entropy of a system as the flux of the entropy current through an arbitrary spacelike hypersurface Σ crossing the system,

$$S = - \int_{\Sigma} s^\nu d\Sigma_\nu. \quad (1)$$

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Since the result of the integral does not depend on the choice of Σ , the entropy is a Lorentz scalar, in agreement with [41] and with the microscopic statistical interpretations of the entropy [18, 22, 23, 40, 45]. This enables us to define, starting from a theory which is necessarily local (to ensure causality), the equilibrium thermodynamics of an isolated macroscopic body, which allows to make a contact with statistical mechanics [17].

The long-lasting debate on the definition of the temperature of moving bodies, traditionally called Planck–Ott imbroglio [32], originates at this point. At first [28, 35, 41, 44], the discussion was oriented in the direction of defining a transformation law of the temperature under Lorentz boosts and work in this direction is still ongoing (see [12] for a recent review). Supporters of both Planck’s and Ott’s views agree on the fact that the transformation should involve a Lorentz factor, however the opinions diverge on its position (at the denominator according to Planck [41], at the numerator according to Ott [35]). Nowadays this approach has become a quest for the most natural-looking choice of variables in the differential of the energy of a moving body (see e.g. [2, 37]).

Other authors have approached the problem studying the equilibrium state of two weakly interacting bodies in motion with respect to each other, trying to understand whether a body “*looks hotter or colder*” from the point of view of the other. van Kampen [24] and Israel [18] have argued that in a covariant framework one must consider that the two bodies can exchange both energy and momentum and therefore the result will depend on the exact circumstance of the experiment. Following the same line of thoughts, Bíró, and Ván [3] have shown that, depending on the phenomenological model of heat transfer which is invoked, one can recover Planck’s, Ott’s or Landsberg’s transformation law. Landsberg and Matsas [27, 29] arrive at similar conclusions considering that a moving photon detector in a heat bath of black body radiation might in general measure different temperatures depending on how it averages the energies of the incoming photons with respect to the direction. Finally, Sewell [43] has studied the problem in the context of quantum statistical mechanics, proving that a body cannot satisfy the KMS conditions for different inertial frames in motion with respect to one another, arriving at the conclusion that a body has a well defined temperature only in its rest frame.

In this paper we propose a new view on the Planck–Ott imbroglio. Using simple arguments of thermodynamics and statistical mechanics we show that the zeroth law of thermodynamics plays a fundamental role in the experimental definition of the temperature and that the deep origin of the controversy is in the fact that the operational notion of thermal equilibrium provided by standard thermodynamics (see e.g. [16]) is not covariant. The purpose of our work is to revisit its definition, in a form which is manifestly covariant. Once this new formulation is provided, it will become clear that, in the absence of spontaneously broken symmetries, two bodies are in thermal equilibrium if and only if they are at rest with respect to each other and they have the same rest-frame temperature. Any condition in which two interacting bodies maintain a relative motion with respect to each other for infinite time will then be shown to be a non-ergodic system and should, therefore, be considered a metastable state, for which a complete description in the framework of equilibrium thermodynamics is not guaranteed to exist.

Throughout the paper natural units are employed: $c = \hbar = k_B = 1$. For the flat space-time metric we adopt the signature $(-, +, +, +)$ and we set $\epsilon_{0123} = +1$.

2 Need for a Zeroth Law

In this first section we prove that, in order to provide a unique definition for the temperature of a system exhibiting further constants of motion apart from the energy, it is necessary to specify how a thermometer is supposed to interact with the system. Our aim is to show that the concept of thermal equilibrium appearing in the zeroth law of thermodynamics relies on strict assumptions about this interaction.

2.1 Ambiguity of the Notion of Temperature

Under the condition that the microscopic Hamiltonian is fixed and time-independent, the second principle of thermodynamics implies that the macrostate of an isolated thermodynamic system in equilibrium is in general fully determined once all the constants of motion (fundamental, like e.g. the energy E , or emergent, like e.g. the winding numbers of the superfluid order parameter phases) are assigned. So, it is natural to write the entropy as a function of these variables, namely

$$S = S(E, X_A), \tag{2}$$

where the X_A are l conserved quantities of the dynamics of the system and we introduced the label $A = 1, \dots, l$. Once an equation of state of this form is given, one is naturally tempted, following the standard textbook convention, to introduce the temperature through the equation

$$\frac{1}{T} = \left. \frac{\partial S}{\partial E} \right|_{X_A}. \tag{3}$$

This, however, can lead to an ambiguity. In fact, if the X_A are constants of motion, then this is also true for any set of quantities

$$Y_B = Y_B(E, X_A), \tag{4}$$

with $B = 1, \dots, l$. The Y_B would, thus, constitute an equivalently good choice of variables in the construction of the equation of state, provided that the map $(E, X_A) \longrightarrow (E, Y_B)$ is one-to-one. So if we start from the equation $S = S(E, Y_B)$, then we are naturally lead to identify the temperature as

$$\frac{1}{T'} = \left. \frac{\partial S}{\partial E} \right|_{Y_B} = \frac{1}{T} + \sum_{A=1}^l \left. \frac{\partial S}{\partial X_A} \right|_E \left. \frac{\partial X_A}{\partial E} \right|_{Y_B}, \tag{5}$$

which may be a priori different from the definition of T . So it is clear that we need to provide a more rigorous and general way to define the temperature, which must be invariant under this change of variables in the equation of state.

2.2 The Role of the Thermometer

To identify a unique definition for the temperature we need to introduce the notion of a thermometer. In standard thermodynamics an ideal thermometer is a system which interacts very weakly with the body object of study. Formally, this means that the interaction enables an exchange of heat among the thermometer and the system, but it plays a negligible role in the definition of the thermodynamic quantities and can be neglected in their computation. Thus the total energy E_{tot} and entropy S_{tot} can be written as

$$E_{\text{tot}} = E + E_{\tau} \quad S_{\text{tot}} = S + S_{\tau}, \quad (6)$$

where E_{τ} and S_{τ} are respectively the energy and entropy of the thermometer. We remark that the two conditions of (6) are presented here as *defining* properties of the thermometer. Their validity is necessary in thermodynamics to avoid the risk of producing further ambiguities in the definition of the temperature associated with the need of specifying the interaction or the correlations [34]. In the presence of strong gravitational fields it might become hard to ensure the additivity of the energies, however, since we are working in special relativity, the space-time is flat and it is in principle possible to make the interaction potential between the body and the hypothetical thermometer arbitrarily small.

We impose that the only constant of motion of the ideal thermometer is the energy, so it is described by an equation of state

$$S_{\tau} = S_{\tau}(E_{\tau}) \quad (7)$$

and there is no ambiguity in the definition of its temperature:

$$\frac{1}{T_{\tau}} = \frac{dS_{\tau}}{dE_{\tau}}. \quad (8)$$

After the thermometer is put into contact with the body, they interact exchanging energy until they reach equilibrium. In equilibrium they must have the same temperature, so we can use the final value of T_{τ} as a measure of T .¹

Let us assume that the interaction of the body with the thermometer does not break the conservation of the quantities X_A . Thus E_{tot} and all the X_A are constant during the evolution of the total system body+thermometer towards equilibrium, which is then given by the condition of maximum entropy

¹ An additional property of the thermometer is that its heat capacity is infinitesimal, which implies that the heat exchange does not affect the state of the body relevantly. This requirement, however, does not need to be invoked in our study. In this sense, if we replace the thermometer with its formal opposite, the heat bath, all our analysis is left unchanged.

$$\left. \frac{\partial S_{\text{tot}}}{\partial E_{\tau}} \right|_{E_{\text{tot}}, X_A} = 0, \quad (9)$$

or, equivalently,

$$\frac{1}{T_{\tau}} = \left. \frac{\partial S}{\partial E} \right|_{X_A}. \quad (10)$$

So, under the assumption that the X_A are conserved by the interaction with the thermometer, we find that the thermometer measures T , given in Eq. (3). One can show with analogous calculations that if, on the other hand, the quantities Y_B were conserved by the interaction with the thermometer, then it would measure T' , given by Eq. (5). So, to solve the ambiguity in the definition of the temperature, one needs to find the X_A which are constants of motion not only when the system is isolated, but also when an exchange of energy with a thermometer is enabled.

Note that the X_A and the Y_B are all simultaneously conserved in the interaction with the thermometer only if $Y_B = Y_B(X_A)$, so in the case in which $T = T'$, see Eq. (5). Therefore if the X_A are conserved and $T \neq T'$, there must be at least one variable Y_B whose conservation is broken due to the exchange of energy with the thermometer.

2.3 An Unconventional Thermometer

In standard thermodynamics of continuous media, the issue we presented in the previous subsection does not seem to appear because in this case there is only one independent constant of motion, apart from energy, which is the number of particles N . It is natural to assume that, in the conventional experimental setting in which a thermometer is in contact with a substance, the conservation of N is not broken, providing a unique definition of the temperature as²

$$T = \left(\left. \frac{\partial S}{\partial E} \right|_N \right)^{-1}. \quad (11)$$

However it is possible to imagine unconventional thermometers which, to measure the temperature, need to break the conservation of N , while keeping fixed an other function $Z = Z(E, N)$. In this subsection we make a simple hypothetical example of this kind of thermometer. Our aim is to convince the reader that the ambiguity we described in this section is not merely formal, but represents a real possibility.

Let us consider an ideal gas. If we assume the particle number N to be conserved when the system is isolated (in the absence of the thermometer), the equation of state can be written in the form

² Note that the volume is not a constant of motion, but it is imposed through an external potential. Therefore it is a parameter in the Hamiltonian [26], which is assumed from the beginning of our discussion to be fixed and time-independent. Thus, in our discussion it does not even need to appear in the equation of state and the fact that it is kept constant in the calculation of the temperature is obvious.

$$S = S(E, N) \quad (12)$$

and we can define the temperature T and the chemical potential μ of the gas according to the conventional prescription

$$dS = \frac{1}{T}dE - \frac{\mu}{T}dN. \quad (13)$$

Now let us imagine to put it into contact with a thermometer, with equation of state of the form (7), which can interact with the gas *only* by absorbing (in an annihilation process) or emitting (in a creation process) particles with a given (positive) energy ε . The total system gas+thermometer now is characterised by two constants of motion: the total energy $E_{tot} = E_\tau + E$ and the quantity

$$Z := E - \varepsilon N. \quad (14)$$

Thus we have built a thermometer which breaks the conservation of N , replacing it with the conservation of Z . The condition of maximum entropy with respect to the exchange of energy between the gas and the thermometer now reads

$$\left. \frac{\partial S_{tot}}{\partial E_\tau} \right|_{E_{tot}, Z} = 0, \quad (15)$$

which produces the condition

$$T_\tau = \frac{\varepsilon}{\varepsilon - \mu} T. \quad (16)$$

Therefore we have proven that a thermometer of this kind, in equilibrium, has a temperature which is different from the (standard) one of the gas.

The reader who was skeptical about the possibility of building such a thermometer without invoking the existence of any sort of Maxwell demon which selects the particles with energy ε can see Appendix A for a very simple hypothetical example of this kind of device. There, Eq. (16) is proven again using pure quantum statistical mechanical arguments.

2.4 The Zeroth Law of Thermodynamics

One can immediately convince oneself that the thermometer that measures the temperature T and the one that measures the temperature T' , presented in Section 2.2, cannot be considered to both reach thermal equilibrium with the system. In fact, having different temperatures, they are not in thermal equilibrium with each other and the zeroth law of thermodynamics states that *the condition of thermal equilibrium is an equivalence relation*. Luckily, in the particular case of systems contained in boxes, at rest with respect to each other, the zeroth law, as it is formulated in standard textbooks, offers a simple unambiguous way to select the constants of motion X_A to keep constant in the calculation of the temperature, removing any ambiguity.

The resolution is given by the definition of the concept of thermal equilibrium appearing in the zeroth law: “two systems are said to be in thermal equilibrium if they are linked by a wall permeable only to heat and they do not change over time” [5]. In the original formulation, walls permeable only to heat (WPOH) are fixed walls which allow the exchange of energy, but not of particles. In the context of quantum optics or high energy physics, where the interactions are mediated by particles, one cannot require the impermeability to every kind of particle, otherwise there would be no interaction at all. However we can redefine the WPOHs as fixed walls which allow the exchange of energy, but not of any Noether charge Q_A arising from internal symmetries of the microscopic theory (for a general definition of conserved charge in Quantum Field Theory see [39]).

Therefore if we make the ergodic hypothesis [25, 36], i.e. if we assume that the only constants of motion of a macroscopic system in a box are the energy and the charges Q_A (which can be typically interpreted as linear combinations of particle minus antiparticle numbers), then we can write the entropy as

$$S = S(E, Q_A), \quad (17)$$

and unambiguously define the temperature as

$$T = \left(\frac{\partial S}{\partial E} \Big|_{Q_A} \right)^{-1}. \quad (18)$$

Every thermometer which is in thermal equilibrium with the system must interact in a way to conserve the amount of Q_A in the system. In fact, the charge is fundamentally conserved and cannot flow from the system to the thermometer (or vice versa), because they are separated by a WPOH. Thus every thermometer measures T .

The unconventional thermometer presented in Subsect. 2.3 can never be in thermal equilibrium with the gas. In fact if we try to link it to the gas through a WPOH and we assume the particles to carry a conserved charge (e.g. a positive baryon number), then no exchange of particles is allowed, so no interaction can occur and the two bodies remain isolated with respect to each other.

Note that Noether charges are, by construction, extensive quantities, being the flux over 3D hypersurfaces of conserved Noether currents,

$$Q_A = - \int_{\Sigma} J_A^\nu d\Sigma_\nu \quad \nabla_\nu J_A^\nu = 0, \quad (19)$$

thus Eq. (17) also justifies the usual assumption that the entropy should be written as a function of extensive variables [4]. So the zeroth law plays a fundamental role in thermodynamics, not only because it solves the problem of ambiguity of the temperature, but also because it selects the quantities which must be used as primary variables in the fundamental relation. In fact, without a zeroth law, any constant of motion of the system would be an equally acceptable free variable to be used in the equation of the entropy.

We remark that there are thermodynamic systems, such as superfluids and more in general systems with broken symmetries, which exhibit emergent constants of

motion which are not conserved charges (and might not even be extensive). In these cases, however, the new constants of motion are collective quantities which can only be altered through the action of very strong interactions, while are conserved by every weak perturbation. Therefore they do not introduce any ambiguity in the definition of temperature and thermal equilibrium (see e.g. [14]) and will be ignored in the present discussion, even if in general they should be included in the equation of state [4].

3 Covariant Equation of State of an Ergodic Body

In this section we derive from first principles the covariant equation of state of an isolated system in special relativity. The aim is to show that the Planck–Ott controversy is simply a particular case of the general problem underlined in Subsect. 2.1.

In a standard thermodynamic language, the term *isolated* means that the system does not exchange energy with the environment, but it may exchange momentum, through e.g. the interaction with adiabatic walls. This asymmetry between energy and momentum cannot hold in a covariant formulation of thermodynamics, because we want to require that the physics (and therefore the form of the Hamiltonian) of the system are the same in any reference frame. Therefore by the term *isolated* we now mean that the system does not have any interaction with any external field, which otherwise would select a preferred reference frame. This immediately rules out the possibility to have an externally imposed volume, removing at the root the annoying (and not well defined) problem of the relativistic transformation of the thermodynamic pressure.

In this context, if a system has finite size it must be self-bounded. If, for example, it is a fluid in a box, the box should be regarded as a dynamical part of the system itself, which can be accelerated, deformed or heated (it has an entropy, c.f. with Landau and Lifshitz [26]). This implies that its shape and volume is only an equilibrium property, not a parameter in the Hamiltonian.

3.1 Deriving the Equation of State

Since, by definition, an isolated system does not have any interaction with the environment, its total four-momentum p^ν is necessarily conserved. The norm of the four-momentum is the mass of the system,

$$M = \sqrt{-p^\nu p_\nu}, \quad (20)$$

and we assume it to be different from zero.

For an isolated system also the angular momentum tensor $J^{\nu\rho}$ is conserved, so special relativity provides automatically other 6 constants of motion which a priori should be included as thermodynamics variable. However, if we go to the reference frame in which $p^\nu = M\delta^\nu_0$, we find that [46]

$$J^{\nu\rho} = \begin{bmatrix} 0 & -Mx_{CM}^1 & -Mx_{CM}^2 & -Mx_{CM}^3 \\ Mx_{CM}^1 & 0 & W^3/M & -W^2/M \\ Mx_{CM}^2 & -W^3/M & 0 & W^1/M \\ Mx_{CM}^3 & W^2/M & -W^1/M & 0 \end{bmatrix}, \tag{21}$$

where \mathbf{x}_{CM} is the position of the center of mass (or, more precisely, “center of energy”) of the system, while

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}p^\sigma \tag{22}$$

is the Pauli–Lubanski pseudovector (proportional to the *spin* of the body, intended as the irreducible part of its angular momentum). Using the invariance of the laws of physics under global translations, we can set the origin in a way that $\mathbf{x}_{CM} = 0$, so we have that

$$J^{\nu\rho} = \frac{1}{M^2}\epsilon^{\nu\rho}{}_{\mu\sigma}p^\mu W^\sigma. \tag{23}$$

Thus, considering also that it must be true that

$$W_\nu p^\nu = 0, \tag{24}$$

the physically relevant (in the construction of the equation of state of a body) constants of motion which the conservation of the tensor $J^{\nu\rho}$ introduces are only three out the four components of the Pauli–Lubanski pseudovector.

We call a body *ergodic*, in analogy with Subsect. 2.4, if the remaining constants of motion are just the Noether charges Q_A of the underlying field theory. We assume that these charges transform as scalars under Lorentz transformations. This is the case for normal continuous media, where they can be typically interpreted as particle (minus antiparticle) numbers. Recalling that the physics of the system is invariant under Lorentz transformations and that the entropy is a scalar, the most general equation of state that an ergodic body can have has, therefore, the form

$$S = S(\sqrt{-p^\nu p_\nu}, \sqrt{W^\nu W_\nu}, Q_A). \tag{25}$$

This result could be equivalently proved in the context of a quantum theory as follows: since the von Neumann entropy is invariant under unitary transformations, it is a Poincaré invariant and therefore it can be written as a function of the Poincaré invariants of the theory, which are the arguments appearing in the equation of state given above.

Equation (25) generalizes the formula of Israel [19], valid for static systems, to the case of rotating objects.

3.2 The Planck–Ott Controversy Reinterpreted

Since our aim is to understand only the role of the motion on the definition of the temperature we assume for simplicity that the Pauli–Lubanski pseudovector is zero and

remains such also in the interaction with the thermometer. We also require that no exchange of Noether charges Q_A is allowed between the body and the thermometer, in agreement with the discussion of Subsect. 2.4. Thus all the arguments in the equation of state (25) apart from the first one are fixed and can be ignored, leaving

$$S = S(\sqrt{-p^\nu p_\nu}). \quad (26)$$

Taking the differential of this equation of state we obtain

$$dS = -\beta_\nu dp^\nu, \quad (27)$$

with

$$\beta^\nu = \frac{u^\nu}{T_{RF}}, \quad (28)$$

where we have introduced the *rest-frame temperature* through the equation³

$$\frac{1}{T_{RF}} := \frac{dS}{dM}, \quad (29)$$

and the normalised four-velocity of the center of mass

$$u^\nu := \frac{p^\nu}{M}. \quad (30)$$

Now the origin of the controversy becomes clear: the system, as a result of covariance requirements, is naturally given in the form (2), with $l = 3$. In fact, chosen a reference frame, apart from the energy $E = p^0$, there are other three constants of motion implied by the conservation of the linear momentum. If we choose to write

$$S = S(E, p_j), \quad (31)$$

where from now on $j = 1, 2, 3$, then Eq. (27) implies

$$dS = \frac{u^0}{T_{RF}} dE - \frac{u^j}{T_{RF}} dp_j \quad (32)$$

and we are naturally lead to define the temperature according to Planck's prescription:

$$T = \frac{T_{RF}}{u^0}. \quad (33)$$

If, on the other hand, we choose to write

³ Note that Eq. (29) defines the rest-frame temperature as a manifest scalar. In late times, Einstein suggested that we should identify this scalar with the thermodynamic temperature [30]. This approach is a convenient linguistic solution currently adopted in relativistic hydrodynamics [33], but it does not solve the problem of predicting the equilibrium state of two bodies in motion with respect to each other.

$$S = S(E, v^j), \tag{34}$$

where

$$v^j = \frac{p^j}{p^0} \tag{35}$$

is the three-velocity of the center of mass of the system, then Eq. (27) implies

$$dS = \frac{1}{u^0 T_{RF}} dE - \frac{E u_j}{T_{RF}} dv^j. \tag{36}$$

This would lead to Ott’s prescription for the temperature

$$T' = u^0 T_{RF}. \tag{37}$$

So the ambiguity in the definition of the temperature in covariant thermodynamics is a particular case of the general problem exposed in Subsect. 2.1.

3.3 A Third Option

Note that p_j and v^j are not the only two possible examples of constants of motion we could choose. Since any triplet of independent functions $Z_j = Z_j(E, p_1, p_2, p_3)$ could be put in the equation of state, there might even be definitions in disagreement with both Planck’s and Ott’s prescriptions.

For example, in analogy with the unconventional thermometer proposed in Subsect. 2.3, we can imagine the following situation. Let us assume a photon gas to be the body of which we want to measure the temperature. Chosen a reference frame, we build a hypothetical thermometer which is kept at rest in this frame by an external force⁴ and which can exchange energy only by absorbing or emitting photons with a fixed four-momentum

$$q = (\varepsilon, \varepsilon, 0, 0), \quad \varepsilon > 0. \tag{38}$$

Then, the conserved quantities of the system gas+thermometer are E_{tot} and the triplet

$$Z_1 = p_1 - E \quad Z_2 = p_2 \quad Z_3 = p_3. \tag{39}$$

With calculations which are analogous to those performed in Subsect. 2.3, with the aid of Eq. (32), we can prove that when the equilibrium is reached the thermometer reports a temperature

⁴ Since the external force acts on the thermometer and not on the body, it does not break the covariance of internal Hamiltonian of the body, therefore the equation of state (26) remains valid.

$$T_\tau = \frac{1}{1 - v^2} \frac{T_{RF}}{u^0}, \quad (40)$$

meaning that it measures a Doppler-shifted temperature, which is different from both (33) and (37).

So, we have clarified that the real source of ambiguity is not special relativity by itself, but the conservation of the total linear momentum, which introduces three new constants of motion in the equation of state.

4 Thermometers in Relativity

In Subsect. 2.4 we have shown how the standard formulation of the zeroth law of thermodynamics can be used to solve the ambiguity presented in 2.1. However, that formulation was referring to the case in which the different bodies are linked by a WPOH, which is clearly an external field, whose presence breaks the covariance of the theory and the conservation of the momentum itself. For these reasons we cannot rely on it in our case. Furthermore, the definition of thermometer we introduced in Sect. 2.2 is not well suited for covariant thermodynamics because the relation (7) is not covariant, but defines an object which must be kept at rest in a given frame by the action of an external force, as in the case presented in Subsect. 3.3.

In this section we outline the steps of an ideal measurement of temperature in a context in which the physics of the body and of the thermometer are the same in any reference frame.

4.1 Operational Definition of Temperature

If we assume the thermometer to be a physical object (which does not break the covariance of the theory) with the least amount of macroscopic degrees of freedom possible, we need to impose it to have an equation of state of the form (26):

$$S_\tau = S_\tau(M_\tau), \quad (41)$$

with M_τ the mass of the thermometer, given by

$$M_\tau = \sqrt{-p_\tau^\nu p_{\tau\nu}}, \quad (42)$$

where p_τ^ν is its four-momentum. Let us consider the following idealized experiment:

- A thermometer is prepared with an initial mass and located inside the system in a not specified state of motion (formally, this means that the four-momentum p_τ^ν in the initial conditions is arbitrary)
- A weak interaction between the body and the thermometer is switched on and the whole system evolves towards its equilibrium state (formally, this is obtained maximizing the total entropy of the full system body+thermometer compatibly with the constants of motion, which are assigned in the initial conditions).

- After the state of equilibrium is reached, the interaction is switched off and the thermometer is extracted from the body. An experimenter, then, weighs the thermometer accurately, without altering its internal state, obtaining a measure of its mass and therefore, using (41), of its (rest-frame) temperature:

$$T_\tau = \left(\frac{dS_\tau}{dM_\tau} \right)^{-1}. \tag{43}$$

The experiment we presented constitutes the ideal temperature measurement procedure in special relativity. Note that the experimental setting presented in Subsect. 3.3 is naturally ruled out in a fully covariant context. In fact the thermometer, absorbing photons, exchanges momentum with the gas, and therefore should accelerate. The only way to prevent it is to admit the existence of an external force which counteracts this effect, breaking the covariance of the theory.

4.2 The Constants of Motion

In the experimental procedure outlined in the previous subsection, the equilibrium state that the isolated system body+thermometer reaches at the end of a transient is the one which maximizes

$$S_{\text{tot}} = S(\sqrt{-p^\nu p_\nu}) + S_\tau(\sqrt{-p_\tau^\nu p_{\tau\nu}}). \tag{44}$$

Thus, we need to impose, see Eq. (28),

$$\delta S_{\text{tot}} = -\beta_\nu \delta p^\nu - \beta_{\tau\nu} \delta p_\tau^\nu = 0 \tag{45}$$

for every perturbation of the eight variables p^ν , p_τ^ν allowed by the conservation of the constants of motion. The conservation of the total four-momentum

$$p_{\text{tot}}^\nu = p^\nu + p_\tau^\nu, \tag{46}$$

however, implies that

$$\delta p^\nu = -\delta p_\tau^\nu, \tag{47}$$

so Eq. (45) becomes

$$(\beta_\nu - \beta_{\tau\nu}) \delta p_\tau^\nu = 0. \tag{48}$$

Thus, we immediately obtain that if the thermometer is able to exchange freely energy and momentum with the body, then the variations δp_τ^ν are completely free and Eq. (48) implies

$$u_\tau^\nu = u^\nu \quad T_\tau = T_{RF}. \tag{49}$$

The first equation is the thermodynamic explanation of the friction as an entropic force. If the two bodies are free to exchange both energy and momentum, the

exchange will lead them to a final state in which they comove and have the same rest-frame temperature.

The only way to avoid the trivial outcome of a thermometer comoving with the body is to impose the presence of other constants of motion, which can force the existence of relative motions also in equilibrium. The easiest way to do this consists of imposing that the four-velocity u_τ^ν of the thermometer is conserved in the interaction with the body (as has been proposed by Becattini [2]). If this is the case, then the variation δp_τ^ν in (48) are no more completely free, but satisfy

$$\delta p_\tau^\nu = u_\tau^\nu \delta M_\tau. \quad (50)$$

Therefore, the condition for the entropy to be maximum with respect to M_τ , with p_{tot}^ν and u_τ^ν constant, reads

$$\frac{u_\nu u_\tau^\nu}{T_{RF}} + \frac{1}{T_\tau} = 0, \quad (51)$$

which in the rest frame of the thermometer becomes Planck's prescription, Eq. (33). However, this is only one possibility. One could instead design a situation in which the thermometer does not alter the state of motion of the body. In this case the constants of motion would be the components of u^ν , giving

$$\delta p_\tau^\nu = -u^\nu \delta M. \quad (52)$$

The final condition would read

$$\frac{1}{T_{RF}} + \frac{u_\nu u_\tau^\nu}{T_\tau} = 0, \quad (53)$$

in agreement with Ott's prescription, Eq. (37). There are, clearly, other possible assumptions for δp_τ^ν which lead to alternative equilibrium conditions.

We have shown that the result of a temperature measurement can vary from thermometer to thermometer if we admit the presence of a relative motion in equilibrium. Everything depends on which energy-momentum exchanges are allowed by the interaction and this, in turn, depends on how the thermometer is designed. This formalizes from a statistical perspective the results of Bíró and Ván [3], who have shown that different phenomenological models of heat transfer (i.e. different directions of allowed four-momentum exchanges) lead to different equilibrium states.

It is, however, clear that a constraint of the kind (50) or (52) represents a non-ergodicity of the system body+thermometer which is unlikely to arise in realistic situations where a form of friction, even if small, always exists.⁵ For the interested reader, we propose in Appendix B an ideal system which seems, at first, to allow for

⁵ Again we remark that in this work systems with spontaneously broken symmetries, like superfluids, are not considered because the role of their emergent constants of motion in the definition of the temperature has already been clarified. The systems under consideration are only the ergodic bodies presented in Subsect. 3.

a permanent relative motion in equilibrium. Then we show how nature finds a way to prevent its existence.

4.3 Indirect Thermometers

Motivated by the work of Landsberg and Matsas [29], we need to clarify a point before moving on with the general discussion. In the context of thermodynamics, a rigorously defined thermometer is an object which always reports only its own temperature. Its operating mechanism is based on the zeroth law and not on the knowledge of the microphysics of the body. To clarify the importance of this distinction we give the following example.

Let us suppose that we need to measure the temperature of a photon gas in equilibrium, at rest with respect to us. Since the number of photons is not conserved, all the local properties of the gas depend on a single parameter. This implies that the average value of any intensive observable can be written as a function of the temperature. The famous equation of state of the radiation gas

$$\varepsilon = a T^4, \quad (54)$$

where ε is the energy density and a is the radiation constant, is an example. As a result, any device which measures an observable whose average value is monotonic in the temperature can be used as an indirect thermometer. We take as an example a hypothetical instrument which measures the energy density and then is calibrated to report on a screen the quantity

$$T_m := (\varepsilon/a)^{1/4}. \quad (55)$$

Clearly, if Eq. (54) holds, we have $T_m = T$, therefore this seems to be a reliable thermometer. Now let us imagine that the conditions in the environment change and suddenly all the interactions conserve the number of photons. Now the gas can have in principle a finite chemical potential μ and will have an equation of state

$$\varepsilon = \varepsilon(T, \mu). \quad (56)$$

If now we try to measure the temperature with the device we presented above we may get a wrong answer, because of the additional dependence of ε on the chemical potential.

So we have shown that there is no guarantee that a device which is calibrated to associate a temperature to the mean value of a different observable will give the correct result when unexpected constants of motion arise. This is the explanation of the non-uniqueness of the outcome of the temperature measurement proposed by Landsberg and Matsas [29] in their thought experiment. In fact, they consider a detector which is calibrated to associate a temperature to the spectral properties of the photon gas at rest and they imagine to use it in a situation in which the total momentum is conserved and different from zero.

In this work we are not dealing with this kind of devices, but only with the ideal thermometers we presented in Subsect. 4.1.

5 Covariant Formulation of the Zeroth Law

We are finally able to revisit the concept of thermal equilibrium, presented in Subsect. 2.4, and adapt it to a relativistic context. As a result, we will also show that it is achieved only when the bodies are at rest with respect to each other.

5.1 A New Notion of Thermal Equilibrium

Since any kind of externally imposed wall breaks the covariance of the theory, we need to revisit the notion of thermal equilibrium replacing the WPOH with a more general interaction. From a microscopic perspective, the wall only permeable to heat is an idealization invoked to describe a condition in which the two systems are linked by a very weak interaction which does not allow the exchange of conserved charges (of the kind described in Subsect. 2.4) among the bodies. Formally, this means that the total Hamiltonian H is the sum of the Hamiltonians of the individual systems, plus a small interaction potential V_I such that

$$\left[Q_A^{(i)}, V_I \right] = 0 \quad \forall A, i, \quad (57)$$

where $Q_A^{(i)}$ is the amount of conserved charge A contained in the system i . To make this theory covariant and compatible at a fundamental level with an underlying Quantum Field Theory, we can require the interaction to have a form

$$V_I = - \int \mathcal{L}_I d_3x, \quad (58)$$

where \mathcal{L}_I is a scalar and constitutes the interaction term of the Lagrangian density (function of the fields and not of their derivatives). In general, the term \mathcal{L}_I will be a coupling of the two bodies with some particles which are exchanged by the systems and therefore mediate the interaction. It is, then, clear that the condition (57) is fulfilled, provided that the mediator does not carry any conserved charge. Thus we can formulate the covariant notion of thermal equilibrium in special relativity as follows: “*two systems are said to be in thermal equilibrium if they weakly interact through charge-neutral (in the sense of Noether’s theorem) mediators and there is an inertial frame in which they are both in a stationary state*”.

Note that, even if this new definition has been derived in a fully covariant context, it can still be applied to those cases in which external agents (like walls) are included. In this sense it represents only a generalization of the previous one, able to include the problem of the moving objects. In fact an ideal wall only permeable to heat is formally described as a surface which acts as a perfectly reflecting wall for baryons and leptons, but which is permeable to photon radiation (a photon does not carry any charge, being its own antiparticle).

5.2 The Zeroth Law in Special Relativity

Now that an extended notion of thermal equilibrium has been proposed, we are able to address from a statistical perspective the question about whether or not the zeroth law can hold also in a fully covariant context.

When two bodies exchange a mediator, this will transfer its four-momentum, which, if the sender body is macroscopic, will be in general randomly oriented and its frequency will take values in a continuous spectrum. Therefore we can conclude that in equilibrium the entropy is maximized with respect to any infinitesimal exchange of four-momentum. Formally this means that we need to take δp_τ^v arbitrary in Eq. (48), leading to Eq. (49), which we present again here for two arbitrary bodies a and b :

$$u_a^v = u_b^v \quad T_{RF}^a = T_{RF}^b. \quad (59)$$

If now we go to the common rest-frame of the two bodies, they are both in a stationary state, proving that indeed they are in thermal equilibrium, according to the definition proposed in the foregoing subsection.

It is in principle not impossible to design ad hoc interactions which make one of the bodies opaque only to one specific value of four-momentum of the mediator and transparent to all the others, producing a constraint on the exchange of four-momentum δp_τ^v in Eq. (48). However, considering that at a fundamental level the interaction is not arbitrary, but has the form (58), we can assume that in realistic situations, if we take into account the processes at all the orders, this phenomenon will be in the end broken (in agreement with the thought experiment proposed in Appendix B). As we explained in Subsect. 4.2, this assumption corresponds to a statement of non-existence of additional constants of motion apart from the Noether charges of the theory. We can, thus, consider it a manifestation of the ergodic hypothesis.

We have shown that in covariant thermodynamics two bodies are in full thermal equilibrium if they have the same rest-frame temperature and are at rest with respect to each other. This immediately implies that the condition of being in thermal equilibrium is still an equivalence relation, whose equivalence classes are all the sets of bodies with the same $\beta^v = T_{RF}^{-1} u^v$. The zeroth law of thermodynamics is, then, still valid.

6 Applications

Below we propose three simple applications of our study.

6.1 Heat Baths in Special Relativity and the Covariant Free-Energy Principle

A heat bath is a body whose macroscopic properties are not relevantly altered by the interaction with the system object of study. In standard thermodynamics this is ensured by imposing an infinite heat capacity. It is clear that in special relativity, when both exchanges of energy and momentum are included, we need to provide a specification also of its inertia. We therefore define a heat bath as body with (effectively) infinite heat capacity and mass. Its equation of state needs to be expressed in a covariant form

as a relationship between its mass M_H and entropy S_H (see Eq. (26)), therefore we may write an expanded effective fundamental relation

$$S_H = S_{H0} + \frac{M_H}{T_H}, \quad (60)$$

where S_{H0} and T_H are two constants. The quantity T_H is the rest-frame temperature and the fact that it is a constant guarantees the divergence of the heat capacity.

When an interaction with a system (whose thermodynamic variables will not carry any label, as usual) is enabled, system and heat bath will exchange energy and momentum, conserving the total four-momentum

$$p_{tot}^\nu = p_H^\nu + p^\nu. \quad (61)$$

Thus, considering that

$$p_H^\nu = M_H u_H^\nu, \quad (62)$$

where u_H^ν is the center of mass four-velocity of the bath, we obtain that

$$\delta u_H^\nu = -\frac{\delta p^\nu + u_H^\nu \delta M_H}{M_H}. \quad (63)$$

In the limit in which $M_H \rightarrow +\infty$, with finite exchanges of four-momentum,

$$\delta u_H^\nu \rightarrow 0, \quad (64)$$

so the heat bath does not accelerate. As a result, a heat bath in special relativity is an object with constant temperature and center of mass four-velocity. The manifestly covariant formulation of this statement is

$$\beta_H^\nu = \frac{u_H^\nu}{T_H} = \text{const.} \quad (65)$$

Let us study the condition of equilibrium of a system in contact with a heat bath. The law of non-decreasing entropy reads

$$\delta S_{tot} = \delta S_H + \delta S \geq 0. \quad (66)$$

With the aid of Eq. (60) we can rewrite this condition as

$$\delta(T_H S_{tot}) = -u_{H\nu} \delta p_H^\nu + T_H \delta S \geq 0. \quad (67)$$

Invoking the conservation of the total four-momentum we find

$$\delta(T_H S_{tot}) = u_{H\nu} \delta p^\nu + T_H \delta S \geq 0. \quad (68)$$

Since $u_{H\nu}$ and T_H are constant, they can be brought inside the variation, producing the condition

$$\delta F \leq 0, \quad (69)$$

with

$$F = -u_{H\nu}p^\nu - T_H S. \quad (70)$$

This is the covariant formulation of the principle of minimum free energy for a system in contact with a heat bath. Note that in the reference frame of the bath the expression for the covariant free energy reduces to

$$F = E - T_H S. \quad (71)$$

Thus we have proven that the equilibrium state of a system in contact with a heat bath is the one which minimizes the Helmholtz free energy (over the manifold of states with $T_{RF} = T_H$) measured in the rest-frame of the heat bath.

This is in agreement with the quantum statistical result of Sewell [43], who proved that a body that serves as a heat bath (in the sense of the zeroth law) can obey the KMS conditions only in its own rest-frame, making Eq. (70) the only possible covariant generalization of the free energy.

Finally we note that if we search for the equilibrium state of the system imposing

$$\delta F = 0, \quad (72)$$

we obtain the conditions

$$u^\nu = u_H^\nu \quad T_{RF} = T_H, \quad (73)$$

in accordance with our formulation of the zeroth law.

6.2 Relativistic Unification of Heat and Friction

The root of the Planck–Ott imbroglio was the disagreement on the relativistic transformation of the heat [12]. The two views, and a simple way of deriving them, are summarized in Appendix C. Now we are able to clarify this issue.

In standard thermodynamics, the work exerted on a system is its change of energy due the time-dependence of external macroscopic forces. Following Landau and Lifshitz [26], a macroscopic force can be modelled as an external field which appears in the microscopic Hamiltonian of the system and the work it makes is the result of a time-dependence of it. In fact, if this field changes with time, this breaks the invariance of the microscopic theory under time translations and, as a result, the (Noether) conservation of the energy of the body.

If we follow this interpretation, the generalization to special relativity is straightforward. Let us consider a system in contact with a heat bath and interacting with an external field (which does not act on the bath directly). From (61) we find

$$\delta p_{tot}^\nu = \delta p_H^\nu + \delta p^\nu. \quad (74)$$

The presence of the field now breaks the invariance of the theory under the whole Poincaré group, because the field might have an arbitrary dependence both on space and time. Therefore the total four-momentum is not conserved and we call its variation work four-vector:

$$\delta\mathcal{W}^\nu := \delta p_{tot}^\nu \neq 0. \quad (75)$$

Note that if we consider the whole universe (external field included) as a bigger system, and the field becomes a dynamical variable itself, then the Poincaré invariance is restored and the four-momentum of the universe is again conserved. Thus $-\delta\mathcal{W}^\nu$ represents the four-momentum transferred from the body to the external environment mediated by the action of the field, in agreement with the common practical interpretations.

Considering that the heat bath does not interact directly with the field, its variation of four-momentum can only be the outcome of a transfer to the body, which allows the introduction of a heat four-vector

$$\delta\mathcal{Q}^\nu := -\delta p_H^\nu. \quad (76)$$

As a result, Eq. (74) becomes the first law in special relativity:

$$\delta p^\nu = \delta\mathcal{Q}^\nu + \delta\mathcal{W}^\nu. \quad (77)$$

Formally, this four-vectorial construction reminds the one proposed by Ott (see Appendix C), but the separation into contributions is completely different. It is interesting to study two particular cases which will clarify its physical meaning and show its consistency with the standard definitions.

Let us consider the case in which no interaction with the heat bath occurs. From (76) we have $\delta\mathcal{Q}^\nu = 0$. Now if we choose a reference frame and decompose (77) into its time and space components we obtain the equations

$$\delta E = \delta\mathcal{W}^0 \quad \delta p^j = \delta\mathcal{W}^j. \quad (78)$$

Let us assume that, in this reference frame, the external field does not depend on time. Then the invariance under time translations is restored and the energy of the body is conserved, giving $\delta E = \delta\mathcal{W}^0 = 0$. However, if the field depends on space, the momentum is not conserved and in principle $\delta\mathcal{W}^j \neq 0$. This is the case, for example, of a gas enclosed in a box with perfectly reflecting walls. These ideal walls are nothing but an external stationary potential which break the conservation of the momentum of the gas. If, on the other hand, the walls start moving slowly, then also the energy can change, and it does according to the well known formula $\delta E = -P\delta V$, where P and V are respectively pressure and volume. Thus in this case we have

$$\delta\mathcal{W}^0 = -P\delta V. \quad (79)$$

So we hope we have convinced the reader that the four-vector $\delta\mathcal{W}^\nu$ is the natural relativistic generalization of the thermodynamic work.

Now we can focus on the heat. To do this we consider an opposite situation in which no external field is applied and the body interacts only with a heat bath. Then $\delta\mathcal{W}^\nu = 0$ and we have

$$\delta p^\nu = \delta\mathcal{Q}^\nu. \quad (80)$$

To study the physical interpretation of this equation, we work in the reference frame of the heat bath and write the decomposition

$$\delta E = \delta Q^0 \quad \delta p^j = \delta Q^j. \tag{81}$$

In this frame the condition of non-increasing free energy (69) can be easily rewritten in the form

$$\left(1 - \frac{T_H}{T_{RF}} u^0\right) \delta E + \frac{T_H}{T_{RF}} u_j \delta p^j \leq 0. \tag{82}$$

If we impose that the body is at rest with respect to the bath, then we have $u^j = 0$ and Eq. (82), combined with (81), becomes

$$\left(1 - \frac{T_H}{T_{RF}}\right) \delta Q^0 \leq 0. \tag{83}$$

This allows us to identify δQ^0 with the heat exchange and we have recovered the Clausius formulation of the second law⁶, which holds strictly for bodies which are at rest with respect to each other. On the other hand, if we impose that the two bodies have the same temperature $T_{RF} = T_H$ and their relative speed is non-zero, but small compared to the speed of light (i.e. $u^0 \approx 1$), we find from the combination of (82) and (81) the condition

$$u_j \delta Q^j \leq 0. \tag{84}$$

However, the only force which acts on a body which is moving with respect to the environment, in the absence of external fields, is the friction with the environment f_F^j . Thus from Newton’s second law we find

$$\delta p^j = f_F^j \delta t, \tag{85}$$

where δt is the time interval in which the exchange of momentum occurs. Plugging (85) into (84) we obtain the other familiar law

$$u_j f_F^j \leq 0, \tag{86}$$

which states that the friction always acts in a way to reduce the relative speed between the bodies (in agreement with our formulation of the zeroth law).

The analysis we have made shows that we can interpret the heat four-vector δQ^ν as the joint action of friction and heat exchanges with the bath. Special relativity has dictated the fundamental non-divisibility of these two dissipative phenomena, as different components of the same four-vector. Our analysis in Subsect. 5.1 also provides a microscopic interpretation of this unification. It shows that, since any

⁶ “There is no thermodynamic transformation whose sole effect is to extract a quantity of heat from a colder reservoir and to deliver it to a hotter reservoir” (see e.g. Huang [17])

interaction is the result of the exchange of a mediator, it always involves a transfer of both energy and momentum, making any separation of the contributions merely artificial.

In conclusion, heat transforms as a four-vector in special relativity. However, its space components do not necessarily vanish in the rest-frame of the body, thus Ott's transformation law does not apply in general.

6.3 Temperature of Rotating Objects

It is well known that rotating relativistic objects in full thermodynamic equilibrium have non-uniform temperature [9]. The Killing condition for the temperature vector field implies that the local rest-frame temperature of the volume elements has a dependence

$$T(R) = \frac{T(0)}{\sqrt{1 - \omega^2 R^2}}, \quad (87)$$

where ω is the (uniform) angular velocity of the body and R is the distance from the axis of rotation. The natural question which now arises is how we can define the temperature of a rotating object which is compatible with the zeroth law. To address this issue we need to consider Eq. (25) (ignoring the presence of charges) including a non-zero norm

$$W = \sqrt{W_\nu W^\nu} \quad (88)$$

of the Pauli–Lubanski pseudovector. The equation of state then reads

$$S = S(M, W). \quad (89)$$

Our intuition tells us that we can define the inverse of the temperature as the partial derivative of S with respect to M , but again we face the problem that we do not know which variables we should keep constant. Thus, in analogy with what we did in the previous sections, we consider an interaction with an ideal (spin-less) thermometer and we search for the maximum of the function

$$S_{tot} = S(M, W) + S_\tau(M_\tau). \quad (90)$$

The maximum needs to be computed compatibly with the constraints of conservation of the total four-momentum and angular momentum tensor:

$$\begin{aligned} p_{tot}^\nu &= p^\nu + p_\tau^\nu \\ J_{tot}^{\nu\rho} &= J^{\nu\rho} + J_\tau^{\nu\rho}. \end{aligned} \quad (91)$$

If one performs the calculation of the variations explicitly, imposing

$$\delta S_{tot} = 0 \quad \delta p^\nu = -\delta p_\tau^\nu \quad \delta J^{\nu\rho} = -\delta J_\tau^{\nu\rho}, \quad (92)$$

they will find, as a unique result, that the thermometer has to be at rest in the center of mass of the rotating body, measuring a temperature

$$T_\tau = \left(\frac{\partial S}{\partial M} \Big|_{W/M} \right)^{-1}. \tag{93}$$

It turns out, however, that this is not the maximum of the entropy, but only a saddle point, which results from the mathematical symmetries of the problem. The entropy does not admit a maximum, but only a supremum, which can be computed using the following argument.

It is natural to assume that

$$\frac{\partial S}{\partial W} \Big|_M \leq 0, \tag{94}$$

because, as the spin grows at constant mass, we are subtracting energy from the “chaotic” motions, forcing it into the collective rotation. We consider a state in which the thermometer and the body are far away from each other, in the limit in which the center of mass of the thermometer goes to infinity. If we remove the spin-angular momentum from the body entirely, setting

$$W = 0, \tag{95}$$

this needs to be transferred into orbital angular momentum to ensure the overall conservation. However the orbital angular momentum of the thermometer scales as

$$L_\tau \sim R p_\tau, \tag{96}$$

where R is the distance of the thermometer from the chosen pole (which can be set in the center of mass of the body) and p_τ is the spatial transversal component of the momentum. Thus we have that, for $R \rightarrow +\infty$, only a negligible value of p_τ is required to provide enough angular momentum to ensure the conservation of the total. Therefore we have proven that with an infinitesimal variation of the momentum of the thermometer we can always remove the spin angular momentum from the body and impose Eq. (95) without violating the conservation of the total angular momentum.

Considering Eq. (94), we see that this transformation is always favourable and leads to a growth of entropy. Thus we obtain that the supremum entropy is achieved for

$$W = 0 \quad T_\tau = T \quad u_\tau^\nu = u^\nu, \tag{97}$$

which is reached only in the limit in which body and thermometer are infinitely far from each other. This result remarkably shows that, even if a rotating body can be, in principle, in thermodynamic equilibrium, it is impossible to associate a temperature to its state from the zeroth law if $W \neq 0$. This, obviously, does not prevent one from defining a generalised notion of temperature through, e.g., the saddle point Eq. (93). However, assuming this as the temperature, all the machinery of equilibrium thermodynamics is not guaranteed to apply consistently.

7 Conclusions

We have shown that the presence of other constants of motion apart from the energy in a thermodynamic system can lead to a fundamental ambiguity in the definition of the temperature. The Planck-Ott imbroglio has been explained to arise as a direct consequence of this problem, produced by the need of requiring the conservation of the total linear momentum.

We have explained that the zeroth law of thermodynamics plays a crucial role in selecting the appropriate experimental setting which defines a temperature measurement, removing all the ambiguities. However, we have seen that the standard notion of thermal equilibrium invoked in classical thermodynamics refers to experimental conditions which break the covariance of the theory at a fundamental level and therefore has to be revisited to be applicable to the case of moving bodies in special relativity.

We have proposed a new notion of thermal equilibrium which is fully covariant and well suited for applications in both relativistic and non-relativistic contexts (provided that the gravitational field is negligible). Under the ergodic assumption we have proven that, using this definition, the zeroth law of thermodynamics is still valid and full thermal equilibrium is achieved only when all the systems have the same rest-frame temperature and are at rest with respect to each other.

We have, then, applied our results to three selected open problems of relativistic thermodynamics. We have provided a covariant formulation of the notion of a heat bath and of the principle of minimum free energy. The result is in complete agreement with quantum statistical mechanical calculations [43], establishing a direct connection of our approach with microphysics. Then, we have proposed a solution to the long-lasting debate about the relativistic notions of thermodynamic work and heat. Our minimal approach has been to generalize the definition provided in standard textbooks [26] to a relativistic spacetime. This extension was constructed in a way to be rigorously defined at every scale, this making it well suited for both theoretical modeling and practical application. In the end, we studied the problem of defining the temperature of rotating relativistic objects. We found that it is not possible to attribute to them a temperature from the zeroth law. This is in accordance with the fact that, since their temperature is not uniform [9], thermometers located at different distances from the rotation axis measure a different temperature.

This work completes the axiomatisation of the relativistic thermodynamics proposed by Israel [19, 20], solving the major open controversies. The formalism was already fully self-consistent (encompassing the first and the second law) and represents the theoretical ground upon which the modern formulations of relativistic hydrodynamics are constructed. Our consistent implementation of the zeroth law provides a direct contact with statistical mechanics and with the standard formulation of thermodynamics.

The take-home message is that heat and friction are the components of a four-vector, representing respectively the exchange of energy and momentum between two thermodynamic bodies. This unification solves the controversy about the

relativistic transformation of the heat and, at the same time, changes our understanding of the zeroth law. In fact, since relativity treats energy and momentum symmetrically, so the relativistic thermodynamics will do, adding to the condition of equilibrium with respect to energy exchanges other three conditions associated with the exchanges of momentum. This produces in equilibrium four constraints, instead of one, proving that in relativity the zeroth law is still valid and still establishes an equivalence relation. However, the equivalence classes now need to be parametrized using four independent parameters (temperature and velocity of the center of mass) instead of just one.

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Appendix

Two-Level Thermometer

We consider an ideal non-relativistic Fermi gas enclosed in a box with periodic boundary conditions. Let $C_{\mathbf{p},\sigma}$ be the annihilation operator of a particle occupying the single-particle state of (discrete) momentum \mathbf{p} and spin $\sigma = \pm 1/2$. They satisfy the standard anticommutation relations

$$\left\{ C_{\mathbf{p},\sigma}, C_{\mathbf{q},\gamma} \right\} = \left\{ C_{\mathbf{p},\sigma}^\dagger, C_{\mathbf{q},\gamma}^\dagger \right\} = 0 \quad (98)$$

and

$$\{C_{\mathbf{p},\sigma}, C_{\mathbf{q},\gamma}^\dagger\} = \delta_{\mathbf{p},\mathbf{q}}\delta_{\sigma,\gamma}. \quad (99)$$

The Hamiltonian of the gas is

$$H_g = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} C_{\mathbf{p},\sigma}^\dagger C_{\mathbf{p},\sigma}, \quad (100)$$

where $\epsilon_{\mathbf{p}}$ is the single-particle energy of a fermion of momentum \mathbf{p} . The operator number of particles is

$$N = \sum_{\mathbf{p},\sigma} C_{\mathbf{p},\sigma}^\dagger C_{\mathbf{p},\sigma}. \quad (101)$$

We imagine the thermometer to be a two-level system⁷ with energy separation $\epsilon > 0$ and ground-state energy conventionally set to zero. Therefore the Hamiltonian of the isolated thermometer has the form, in an appropriate basis,

$$H_\tau = \epsilon \sigma_+ \sigma_- = \frac{\epsilon}{2}(1 + \sigma_z), \quad (102)$$

where σ_\pm are constructed as

$$\sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) \quad \sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \sigma_-^\dagger, \quad (103)$$

the operators $\sigma_x, \sigma_y, \sigma_z$ being the Pauli matrices acting on the two-dimensional state space of the thermometer. The total Hilbert space is assumed to be the tensor product between the space of the thermometer and the Fock space of the fermions and we have

$$\left[\sigma_j, C_{\mathbf{p},\sigma} \right] = 0. \quad (104)$$

When the thermometer is put into contact with the gas, we assume the total Hamiltonian to be

$$H = H_g + H_\tau + V_{g\tau}, \quad (105)$$

where the interaction term $V_{g\tau}$ has the form

$$V_{g\tau} = \sum_{\mathbf{p},\sigma} g_{\mathbf{p}} \left[\sigma_+ C_{\mathbf{p},\sigma} + C_{\mathbf{p},\sigma}^\dagger \sigma_- \right], \quad (106)$$

where $g_{\mathbf{p}}$ are real coefficients (reassuring Hermitianity). This potential models a process in which the thermometer makes level transitions absorbing and destroying (or

⁷ We consider a single two-level system for simplicity, but the discussion can be easily generalized to a large number of two-level systems in the thermodynamic limit, producing a rigorously macroscopic object.

creating and emitting) a fermion, breaking the particle number conservation. However it is easy to check that the operator

$$Q = N + \sigma_+ \sigma_- \tag{107}$$

commutes with H and therefore describes a constant of motion. In the limit of small $V_{g\tau}$ (weak interaction) the quantity Z introduced in Eq. (14) is represented by the operator

$$Z = H - \varepsilon Q, \tag{108}$$

and is, thus, conserved.

When the system gas+thermometer is in thermal equilibrium, its density matrix can be assumed to be the Generalised Gibbs ensemble [38]

$$\rho \propto \exp(-\beta H + \lambda Q). \tag{109}$$

In the limit $V_{g\tau} \rightarrow 0$ it splits into the tensor product

$$\rho = \rho_g \otimes \rho_\tau, \tag{110}$$

with

$$\begin{aligned} \rho_g &\propto \exp(-\beta H_g + \lambda N) \\ \rho_\tau &\propto \exp(-\beta H_\tau + \lambda \sigma_+ \sigma_-). \end{aligned} \tag{111}$$

The state ρ_g is the thermal state of the Fermi gas with temperature T_g and chemical potential μ given by

$$T_g = \beta^{-1} \quad \mu = \lambda \beta^{-1}. \tag{112}$$

On the other hand, ρ_τ is the thermal state of the thermometer with temperature

$$T_\tau = \left(\beta - \frac{\lambda}{\varepsilon} \right)^{-1}. \tag{113}$$

Combining (112) with (113) we find

$$T_\tau = \frac{\varepsilon}{\varepsilon - \mu} T_g, \tag{114}$$

which is in agreement with Eq. (16). Note that, since the gas is a Fermi gas, μ can in principle be positive. So T_τ can be infinite (for $\mu = \varepsilon$) or negative (for $\mu > \varepsilon$). In the diluted limit (for $\mu \rightarrow -\infty$) the temperature of the thermometer goes to zero, because the number of fermions is small compared to the number of available single particle states, so the probability to have a particle with exactly the energy ε to be absorbed is small.

We remark that having $T_\tau \neq T_g$ is possible only as a result of the existence of an other constant of motion apart from the energy, as we clarified in Sect. 2. In fact, if we break the conservation of Q , then $\lambda = 0$ (so the chemical potential μ is in turn zero) and (114) becomes $T_\tau = T_g$.

A Problem of Ergodicity

We have proven that a thermometer can maintain, in equilibrium, a relative motion with the body only if we impose some limitations on the possible exchanges of four-momentum allowed by the interaction, preventing δp_r^ν from having arbitrary direction. It is clear that, for macroscopic (i.e. comprised of large numbers of particles) systems with realistic interactions, the existence of an *exact* constant of motion of this kind is nearly impossible to occur. We propose the following thought experiment to explain the basic mechanisms which prevent it.

Measuring Planck's and Ott's Temperature: An Attempt

Consider a real scalar boson ϕ of mass m , which self-interacts with a short-range (compared with the average distance among the bosons) interaction. Since this particle coincides with its own antiparticle we can assume reactions of the type

$$\phi + \phi \rightleftharpoons \phi + \phi + \phi \quad (115)$$

to be allowed, implying that the particle number is not conserved (the chemical potential of the boson vanishes). There are, therefore, no conserved charges and the thermal state of this scalar boson can be characterised by an equation of state of the type (26). We will consider a gas of bosons ϕ to be the body of which we measure the temperature.

We construct the thermometer as a collection of comoving, with respect to each other, distinguishable two-level systems, where the ground state is denoted by $|0\rangle$ and the excited state is denoted by $|1\rangle$. We impose the mass separation between the two levels to be equal to the mass of the boson:

$$M_{|1\rangle} - M_{|0\rangle} = m. \quad (116)$$

The interaction between the thermometer and the gas is assumed to happen only through the absorption or the emission of a boson ϕ by a two-level system, namely through a reaction

$$B_{|0\rangle} + \phi \rightleftharpoons B_{|1\rangle} \quad (117)$$

where $B_{|0\rangle}$ represents a two-level system in the ground state and $B_{|1\rangle}$ represents a two-level system in the excited state. Equation (116) implies that the reaction (117) can happen only in the case in which the boson ϕ is at rest with respect to the thermometer, to conserve the total four-momentum. Therefore the reaction alters the mass of the thermometer but not its state of motion. We have constructed a thermometer whose four-velocity u_r^ν is conserved by the interaction with the body.

According to the statistical arguments we exposed in Subsect. 4, this thermometer will measure, in equilibrium, Planck's temperature. This can be also shown directly from kinetic theory. In fact, if we work in the reference frame of the thermometer, it is easy to show that the number $N_{|1\rangle}$ of two-level systems in the excited state evolves according to the Eq. [11]

$$\frac{dN_{|1\rangle}}{dt} = \mathcal{R}[N_{|0\rangle}n_0 - N_{|1\rangle}(1 + n_0)], \quad (118)$$

where \mathcal{R} is the spontaneous decay rate of the state $|1\rangle$, $N_{|0\rangle}$ is the number of two-level systems in the ground state and $n_{\mathbf{q}}$ is the average number of bosons ϕ with spatial momentum \mathbf{q} . Once thermal equilibrium is reached, $N_{|1\rangle}$ becomes constant and from (118) we obtain

$$\frac{N_{|1\rangle}}{N_{|0\rangle}} = \frac{n_0}{1 + n_0}. \quad (119)$$

Recalling that the interaction is short range, we can approximate the gas as ideal. The equilibrium occupation number $n_{\mathbf{q}}$ of the ideal boson gas in a thermal state is

$$n_{\mathbf{q}} = \frac{1}{e^{-\beta_{\nu} q^{\nu}} - 1}, \quad (120)$$

thus equation (119) becomes

$$\frac{N_{|1\rangle}}{N_{|0\rangle}} = e^{-\beta^0 m}, \quad (121)$$

which describes the macrostate of the thermometer having temperature

$$T_{\tau} = \frac{1}{\beta^0}. \quad (122)$$

Recalling Eq. (28), we see that the above equilibrium condition reduces to Planck's prescription. Thus, this thought experiment might seem, at the first sight, to support Eq. (33) as the transformation law of the temperature under Lorentz boosts. A supporter of Ott's view, however, performing the same experiment might consider the collection of two-level systems to be body of which they are measuring the temperature and the boson gas as the thermometer. They will therefore exchange T_{τ} with T_{RF} , obtaining a measurement of the temperature of the two-level system in agreement with Ott's law, Eq. (37).

Realistic Interactions

The situation we have proposed represents an idealised system in which the thermometer can exchange energy but not momentum (in its reference frame). Unfortunately such a system cannot be constructed in realistic conditions and represents only a non-physical limit. In fact the spontaneous decay rate \mathcal{R} appearing in Eq. (118) is given by the *Fermi Golden Rule* and must be equal to [39]

$$\mathcal{R} = \frac{|\mathbf{q}_f|}{8\pi M_{|1\rangle}^2} |\langle 0, \phi | V_I | 1 \rangle|^2, \quad (123)$$

where $|\langle 0, \phi | V | 1 \rangle|^2$ is the matrix element of the interaction potential responsible for the transition and \mathbf{q}_f is the momentum of the emitted particle ϕ in the original frame of the two-level system. So if we set $\mathbf{q}_f = 0$, which is the necessary condition to guarantee the conservation of u_τ^ν , we will find that the reaction cannot occur and the thermometer does not interact with the body at all. The only way to make the transfusion of energy possible is to set $|\mathbf{q}_f| \neq 0$, breaking the conservation of u_τ^ν . The typical acceleration that the thermometer will, then, experience is

$$a_c \sim \mathcal{R} \frac{|\mathbf{q}_f|}{M_{|0\rangle}}, \quad (124)$$

which is a second order in $|\mathbf{q}_f|/M_{|0\rangle}$. Therefore the system presented in Subject. B.1 can exist only as a limit in which the time required for the thermometer to reach comotion with the body is much longer than the time of the experiment.

There is, however, another complication. Interactions coming from an underlying field theory in general admit the possibility of having higher order processes, of the kind

$$B_{|0\rangle} + \phi + \phi \rightleftharpoons B_{|1\rangle} + \phi, \quad (125)$$

where a spectator ϕ takes part of the four-momentum. In this situation the two-level system does not need to be at rest at the end of the process, breaking again the conservation of u_τ^ν . This process is not suppressed by the phase space and therefore becomes dominant, leading anyway to the inevitable acceleration of the thermometer.

Ambiguity in the Definition of the Heat

Let us start from the general differential

$$\delta p^\nu = u^\nu \delta M + M \delta u^\nu. \quad (126)$$

This represents an orthogonal decomposition of the variation of the four-momentum, because

$$u_\nu \delta u^\nu = 0, \quad (127)$$

which is ensured by the conservation of the normalization condition $u_\nu u^\nu = -1$. Let us assume an equation of state (26), which implies that we do not have chemical-type forms of work $\mu^A \delta Q_A$ (where μ^A is the chemical potential of the charge Q_A), but the only possible way to exert work on the body is by impressing an acceleration through the action of a force. This suggests a relativistic first-law

$$\delta p^\nu = \delta Q^\nu + \delta \mathcal{W}^\nu \quad (128)$$

with

$$\delta Q^{\nu} = u^{\nu} \delta M \quad \delta \mathcal{W}^{\nu} = M \delta u^{\nu}. \quad (129)$$

The four-vector $\delta \mathcal{W}^{\nu}$ can be interpreted as the four-dimensional generalization of the work element. In fact, in a given reference frame,

$$\delta \mathcal{W}^0 = (u^0)^3 M v_j \delta v^j, \quad (130)$$

which is the formula for the relativistic work of an external force acting on a moving particle. From Eq. (129) one finds that in a given frame

$$\delta Q^0 = u^0 \delta M = u^0 T_{RF} \delta S, \quad (131)$$

where we have employed the definition (29). This formula is the relativistic transformation of the heat proposed by Ott [12].

There is, however, another possibility. Let us rewrite Eq. (20) in a chosen reference frame in the form

$$E = \sqrt{M^2 + p_j p^j}. \quad (132)$$

Its differential reads

$$\delta E = \frac{\delta M}{u^0} + v^j \delta p_j. \quad (133)$$

The second term reminds the standard formula of the variation of the kinetic energy provided in Hamiltonian mechanics, which then suggests the decomposition

$$\delta E = \delta Q + \delta \mathcal{W}, \quad (134)$$

with

$$\delta \mathcal{W} = v^j \delta p_j \quad (135)$$

and

$$\delta Q = \frac{\delta M}{u^0} = \frac{T_{RF} \delta S}{u^0}. \quad (136)$$

This is the relativistic transformation for the heat proposed by Planck [12]. We see that the source of ambiguity is in the definition of the work as the variation of the kinetic energy of the system in situations in which the mass is changing with time (in the case in which $\delta M = 0$ the formulas of the work (130) and (135) coincide, as it happens in point-particle mechanics).

As we clarify in the main text, none of these two subdivisions reflects the standard physical interpretation of the splitting of the energy variation into heat and work. The subdivision needs to be performed considering how the environment is acting on the body and cannot be uniquely determined in terms of internal properties of the body itself.

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Chapter 3

Proving the Lorentz invariance of the entropy and the covariance of thermodynamics



Proving the Lorentz Invariance of the Entropy and the Covariance of Thermodynamics

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Abstract

The standard argument for the Lorentz invariance of the thermodynamic entropy in equilibrium is based on the assumption that it is possible to perform an adiabatic transformation whose only outcome is to accelerate a macroscopic body, keeping its rest mass unchanged. The validity of this assumption constitutes the very foundation of relativistic thermodynamics and needs to be tested in greater detail. We show that, indeed, such a transformation is always possible, at least in principle. The only two assumptions invoked in the proof are that there is at least one inertial reference frame in which the second law of thermodynamics is valid and that the microscopic theory describing the internal dynamics of the body is a field theory, with Lorentz invariant Lagrangian density. The proof makes no reference to the connection between entropy and probabilities and is valid both within classical and quantum physics. To avoid any risk of circular reasoning, we do not postulate that the laws of thermodynamics are the same in every reference frame, but we obtain this fact as a direct consequence of the Lorentz invariance of the entropy.

Keywords Thermodynamics · Special relativity · Field theory

1 Introduction

The total thermodynamic entropy S , in equilibrium, must be Lorentz invariant. Every statistical mechanical view on thermodynamics agrees on this point. Whether we identify S with the Boltzmann entropy [1, 2], or with the Gibbs/Shannon entropy [3, 4], or with the von Neumann entropy [5, 6], its Lorentz invariance seems inescapable. This fact is also a foundational feature of relativistic fluid dynamics [7, 8] and of thermal quantum field theory [9].

Intuitively, the invariance of the entropy with respect to Lorentz transformations is usually justified by invoking its statistical connection with microscopic

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probabilities (or numbers of quantum states), which are supposed to have an invariant nature [10, 11]. However, when it comes to proving rigorously, from first principles, that the *thermodynamic* entropy (namely, the macroscopic state function which is subject to the second law) must necessarily be a scalar, some conceptual problems arise and it is easy to fall into circular reasoning.

The thermodynamic argument for the Lorentz invariance of the entropy that is often repeated in the literature [12, 13] is an oversimplified version of an argument originally proposed by [14]. Consider the following thought experiment: a body X is accelerated from being at rest with respect to an observer A to being at rest with respect to an observer B (in motion with respect to A). If the process is adiabatic, it is reversible, hence the entropy of X measured by A is the same before and after the acceleration: $S_i(A) = S_f(A)$. Now let's assume that during this process the rest-frame properties of the body do not change (hence we may call this process a *pure acceleration*). It follows that the initial state, as seen by A , is identical to the final state, as seen by B , which implies $S_i(A) = S_f(B)$ (recall that the entropy is a state function). Thus, A and B agree on the value of the entropy at the end of the process, $S_f(A) = S_f(B)$, proving the Lorentz invariance of the entropy.

The problem with this argument is that what determines whether a process is reversible or not is the difference in entropy between the initial and the final state (if $\delta S = 0$, the process is reversible). Hence, assuming that pure accelerations are reversible is equivalent to assuming that the entropy does not depend on the velocity of the body, which is exactly what we are trying to prove. To the best of our knowledge, the first author who noted this circularity problem was van Kampen [15], who elevated the existence of reversible pure accelerations to the rank of fundamental postulate of relativistic thermodynamics. He showed that no entirely thermodynamic argument can be used to prove *ab initio* the Lorentz invariance of the entropy, but, to set the foundations of covariant thermodynamics rigorously (and to avoid any circularity issue), one only needs to postulate that pure accelerations are reversible.

The goal of the present paper is to explore the validity of van Kampen's postulate in greater detail. In fact, from an operational point of view, the postulate can be rephrased as follows: *adiabatic accelerations (i.e. slow variations of velocity generated by weak mechanical forces) do not alter the rest-frame properties of a body; in particular, they do not affect its rest mass*. Given that this is a simple statement about the behaviour of many-particle systems subject to external forces, it should be possible to test it using relativistic dynamics and quantum field theory.

We remark that the purpose of this paper is not to convince the reader that the entropy is Lorentz invariant; this is already a well established fact [16]. Instead, the aim is to explain why this is the *only* possibility and to prove that any alternative construction of relativistic thermodynamics would lead to serious inconsistencies.

Throughout the paper we adopt the signature $(-, +, +, +)$ and work with natural units $c = k_B = \hbar = 1$. The study is performed within special relativity, hence the metric is assumed flat. Greek space-time indices μ, ν, ρ run from 0 to 3, while latin space indices j, k run from 1 to 3.

2 The Rationale of the Argument

If we want to make our argument solid and unquestionable, we need, first, to understand which assumptions about relativistic thermodynamics we are reasonably allowed to uphold, and which might lead us to circular reasoning.

2.1 Must the Laws of Thermodynamics be the Same in Every Reference Frame?

It is possible to formulate many arguments for the Lorentz invariance of the entropy, based on the assumption that the laws of thermodynamics should be the same in every reference frame. A well-known example is Planck's original argument (which is more refined than the version reported in the introduction), of which we present a slightly more formal version in appendix A. The rationale of Planck's argument [14], and of most of the other thermodynamic arguments present in the literature, is that the entropy is ultimately a rule, which dictates which processes are possible (for thermally isolated systems) and which are not. For example, if a macroscopic state ψ has a lower entropy than a macroscopic state ψ' , this means that, if we keep the system thermally isolated, the process $\psi \rightarrow \psi'$ is possible, while the inverse process is not. Clearly, statements about the possibility for a process to occur cannot depend on the reference frame, hence the entropy must be Lorentz invariant.

The problem with these arguments is that they all treat thermodynamics as a fundamental theory, which should be subject to the principle of relativity in the same way as dynamics is, and whose laws should, therefore, be equally valid in every reference frame. In other words, it is assumed in these arguments that thermodynamics should share the same symmetries of dynamics. However, we already know that there is at least one symmetry for which this is not true: CPT. While CPT is a fundamental symmetry in quantum field theory [17], it is manifestly violated by the second law of thermodynamics. This shows us that we are in general not allowed to treat thermodynamics on the same footing as dynamics.

The fundamental distinction between dynamics and thermodynamics is that dynamics studies the evolution of systems with arbitrary initial conditions, which implies that the solutions of the equations which govern dynamics form a set Λ that is necessarily invariant under the action of the symmetry group \mathcal{G} of the spacetime ($\mathcal{G}\Lambda = \Lambda$). On the other hand, thermodynamics deals only with a subset $\lambda \subset \Lambda$ of solutions, whose initial conditions have precisely those statistical properties (e.g. molecular chaos, see [18]) which give rise to the second law as an emergent quality. It might be the case (and for CPT it is the case!) that these constraints on the initial conditions lead to a symmetry breaking, namely to a situation in which $\mathcal{G}\lambda \neq \lambda$. Considering that specifying the laws of thermodynamics is essentially equivalent to specifying λ , it follows that thermodynamics might in turn not be symmetric under \mathcal{G} .

Let us remark that we are not claiming that the laws of thermodynamics are not Lorentz covariant. They are. But (as we will show in subsection 2.3) their covariance follows from the invariance of the entropy, and not vice-versa. Thus, in a

paper whose goal is to prove the invariance of the entropy, we are not allowed to include the assumption that thermodynamics is the same in every reference frame among the hypotheses.

As a last comment on this issue, we point out that, if one adopts Jaynes' statistical justification for the second law [3], then the initial conditions that give rise to λ are actually the overwhelming majority of initial conditions which are compatible with the initial macroscopic data (in the thermodynamic limit). Hence, it is to be expected that, if the group \mathcal{G} conserves the *causal* ordering of the events (namely if it does not convert initial states into final states), then λ should be approximately invariant under \mathcal{G} . This would explain why thermodynamics is not invariant under CPT (namely $\text{CPT}\lambda \neq \lambda$) while it is expected to be invariant under the proper orthochronous Lorentz group ($SO^+(3, 1)\lambda = \lambda$). In fact, CPT converts initial data into final data, whereas $SO^+(3, 1)$ conserves the causal structure of the field equations by construction [19]. This is the actual statistical justification for the covariance of thermodynamics, because it is not grounded on the interpretation that one chooses to give to the entropy, but on the statistical origin of irreversibility, which constitutes the very foundation of thermodynamics. However, as this argument is qualitative, and thermodynamics does not entirely reduce to Jaynes' view [20, 21], it is important to have also a more formal proof, which is the purpose of the present paper.

2.2 The Assumptions of the Argument

Motivated by the complication outlined in the previous subsection, we need to make an argument for the Lorentz invariance of the entropy which does not build on the assumption that the second law of thermodynamics is valid for every observer. Instead, we will base our argument only two uncontroversial assumptions, namely

- (i) There is a global inertial reference frame A in which it is possible to unambiguously define a notion of entropy S that obeys the second law: $\dot{S} \geq 0$. In this reference frame, bodies may interact with each other, accelerate, decelerate and be destroyed, but the total entropy of isolated systems can never decrease.
- (ii) The microscopic dynamics can be modelled using a field theory, governed by a Lorentz invariant Lagrangian density.

Assumption (i) is simply the requirement that there is *at least* one observer for which the laws of thermodynamics, in their standard "textbook" formulation, are valid. Assumption (ii) is the statement that, although thermodynamics might in principle not admit a covariant formulation, dynamics does. We are enforcing the principle of relativity on the underlying microscopic theory, rather than imposing it directly on thermodynamics.

Throughout the rest of this paper, we will always work in the reference frame A introduced in assumption (i), so that thermodynamics works as usual. In this way we will avoid any possible source of confusion.

2.3 Van Kampen’s Argument

Let us now briefly revisit van Kampen’s argument for the Lorentz invariance of the entropy [15].

We consider an isolated (freely moving) body in thermodynamic equilibrium with total four-momentum p^ν and rest mass $M = \sqrt{-p^\nu p_\nu}$. The entropy in equilibrium must be a function of the constants of motion of the body. To capture the essence of the problem, we assume for simplicity that the only relevant constants of motion are the components of the four-momentum,¹ so that

$$S = S(p^\nu). \tag{1}$$

At this stage, the function $S(p^\nu)$ may be completely arbitrary, because (as we anticipated) we are not excluding a priori the possibility that thermodynamics may break Lorentz covariance. Similarly to what we did in the introduction, let us postulate that it is possible to make infinitesimal *reversible pure accelerations*, namely transformations δp^ν such that

$$\delta S = \frac{\partial S}{\partial p^\nu} \delta p^\nu = 0 \quad (\text{reversible}) \tag{2}$$

and

$$\delta M = -\frac{p_\nu}{M} \delta p^\nu = 0 \quad (\text{pure acceleration}). \tag{3}$$

If these accelerations can have arbitrary direction (i.e. if those δp^ν that satisfy (2) and (3) form a 3D plane), then it follows that there is a function T such that

$$dS = \frac{dM}{T}, \tag{4}$$

which in turn implies

$$S = S(M). \tag{5}$$

The fact that the entropy can be written as a function of a Lorentz scalar implies that, when we perturb the system, the second law of thermodynamics ($\dot{S} \geq 0$) takes the form of a Lorentz-invariant statement:

$$\frac{\dot{M}}{T(M)} \geq 0. \tag{6}$$

¹ The only relevant constants of motion of an ergodic body are four-momentum, angular momentum (tensor) and conserved charges (like the baryon number). If we work at fixed conserved particle numbers, and assume that the body is non-rotating, equation (1) follows. The volume cannot be treated as an independent variable in relativistic thermodynamics. In fact, if a given volume is imposed through external walls, the body is not isolated. Finite isolated bodies are self-bounded, hence their volume is an equilibrium property (like the volume of stars and nuclei) and not a free parameter [22].

But this implies that the set λ of all the initial conditions which realise the second law is invariant under the action of the proper orthochronous Lorentz group (formally, $SO^+(3, 1) \lambda = \lambda$), proving that thermodynamics admits a covariant formulation, in which S is a Lorentz scalar. This sets solid foundations for relativistic thermodynamics.

Our goal, now, is to prove that, if assumptions (i) and (ii), as stated in the previous subsection, are valid, a set of infinitesimal transformations that satisfy both (2) and (3) always exist (at least in principle), converting van Kampen's postulate into a theorem.

3 Reversible Accelerations

Our first task is to understand how we may induce an ideal reversible acceleration on a body. Following [23], the most perfect form of reversible process is an *adiabatic* process, namely an infinitely slow transformation in which the system is kept thermally isolated. Such processes can be modelled, at the microscopic level, as transformations induced by a weak and slow time-dependence of the microscopic Hamiltonian. Our aim is to design an adiabatic transformation which can alter the state of motion of a relativistic body.

3.1 Small Kicks

Let φ_i be the microscopic fields of the body and $\mathcal{L}_{\text{Body}}(\varphi_i, \partial_\mu \varphi_i)$ the Lagrangian density governing the microscopic dynamics. Assume that we are able to generate and control an external potential ϕ (a real scalar field, for simplicity), which interacts with the body through a small dimensionless coupling constant ϵ , so that the action takes the simple form

$$\mathcal{I}[\varphi_i] = \int \left[\mathcal{L}_{\text{Body}}(\varphi_i, \partial_\mu \varphi_i) + \epsilon \phi G(\varphi_i) \right] d^4x, \quad (7)$$

where $G(\varphi_i)$ is an observable. The potential ϕ is an *assigned* real function of the coordinates $\phi(x^\nu)$. It is not a dynamical degree of freedom of the total system ("body + ϕ "), but it plays the role of a source in the action $\mathcal{I}[\varphi_i]$, which breaks the Poincaré invariance of the theory. In a quantum description, the field ϕ plays the role of a classical source [19]; it is not a quantum field. We model ϕ in this way because we want to treat it as a purely mechanical and non-statistical entity (like any other source of thermodynamic work, see e.g. [22]), so its evolution must be completely known and cannot be affected by the statistical fluctuations of the dynamical fields φ_i . In this sense, the potential ϕ may be seen as an analogue of the perfectly reflecting walls of an adiabatic box: it carries no entropy. This implies that the body remains thermally isolated [23] and the second law of thermodynamics holds for the entropy of the body alone [3], also during its interaction with ϕ .

Assume that $\phi = 0$ for $t \leq 0$ (recall that we always work, for clarity, in the reference frame A in which we have a notion of entropy). The configuration of

the system for $t \leq 0$ is the initial state of the body, which is assumed to be an equilibrium state, with four-momentum p^ν . At $t = 0$ we switch on the external potential and we keep it active for a finite time τ , namely

$$\phi \neq 0 \quad \text{for } t \in (0, \tau). \tag{8}$$

No assumption about the duration τ of the process, nor about the exact space-time dependence of $\phi(x^\nu)$, is made. We only require that there is at least a small region of space-time (between the times 0 and τ) in which

$$(\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 > 0, \tag{9}$$

so that we know that the action (7) is not invariant under space translations, breaking the Noether conservation of linear momentum of the body. At the end of the process ($t = \tau$), the four-momentum of the body has changed of a finite amount δp^ν . After some more time passes, the system can reach a new state of equilibrium, whose entropy is $S(p^\nu + \delta p^\nu)$. The total variation of entropy experienced by the system during all this process (including the final relaxation to a new equilibrium) is the finite difference

$$\delta S = S(p^\nu + \delta p^\nu) - S(p^\nu). \tag{10}$$

The aforementioned process may be interpreted as a small kick generated by an ideal mechanical device:

- For $t \leq 0$ the body is completely isolated and in thermodynamic equilibrium. It moves freely across space-time, with initial mass $M = \sqrt{-p^\nu p_\nu}$ and center-of-mass four-velocity $u^\nu = p^\nu/M$. It is in the maximum entropy state possible (as measured in the frame A) compatible with this value of four-momentum.
- For $0 < t < \tau$ the body interacts with a mechanical device with no microscopic degrees of freedom (zero entropy). The interaction is mediated by a potential ϕ , which is generated solely by the device (and therefore carries no entropy). Through this interaction, the body feels a force, which impresses on it a small kick, changing its total four-momentum by an amount δp^ν . This amount of energy and momentum is transferred through ϕ to the device, which is however not explicitly modelled here.
- For $t \geq \tau$ the body is again completely isolated and has time to dissipate all the fluctuations and vibrations induced by the kick, to reach a new equilibrium.

Comparing this description with subsection 6.2 of our previous paper [22], one can see that the variation of four-momentum δp^ν produced in a kick has the nature of pure work (using the terminology we introduced there: $\delta p^\nu = \delta \mathcal{W}^\nu$), because the external agent can be modelled as a purely mechanical entity. Hence, kicks are the simplest form of work-type energy-momentum transfers in relativistic thermodynamics.

3.2 Infinite Infinitesimal Kicks

The key insight which leads us to a notion of adiabatic acceleration is how the changes δp^ν and δS scale with the strength of the coupling constant ϵ , in the limit in which $\epsilon \rightarrow 0$. We take this limit at fixed initial state of the body (for $t \leq 0$) and keep the function $\phi(x^\nu)$ fixed.

Since ϵ quantifies how strongly the system reacts to the presence of the external potential ϕ (ϵ is analogous to the coupling constant q in the electrostatic force $\mathbf{F} = q\mathbf{E}$), it is easy to see that, to the leading order in ϵ , we have the scaling

$$\delta p^\nu \sim \epsilon. \quad (11)$$

However, the variation of the entropy scales differently. In fact, the second law implies $\delta S(\epsilon) \geq 0 \forall \epsilon$. On the other hand, ϵ may have arbitrary sign,² which implies that if we assume $\delta S \sim \epsilon$ we get a contradiction with the second law. Thus, the leading order must be

$$\delta S \sim \epsilon^2, \quad (12)$$

or higher (but even).

Now, consider a sequence of N kicks ($N \rightarrow +\infty$) with a coupling constant $\epsilon = 1/N \rightarrow 0$. The total variation of the four-momentum (due to the whole sequence of kicks) is

$$(\delta p^\nu)_{N \text{ kicks}} \sim N \times (\delta p^\nu)_{1 \text{ kick}} \sim N \times \frac{1}{N} = 1, \quad (13)$$

while the total variation of entropy is

$$(\delta S)_{N \text{ kicks}} \sim N \times (\delta S)_{1 \text{ kick}} \sim N \times \frac{1}{N^2} = \frac{1}{N}. \quad (14)$$

This implies that, as the number of kicks goes to infinity and their intensity goes to zero, the resulting transformation is non-trivial (δp^ν is finite) and reversible ($\delta S = 0$). Hence, we have just built a microscopic model for a reversible acceleration. As expected, it is infinitely slow (duration $\geq \tau \times N \rightarrow +\infty$), so we have rediscovered the well-established fact that adiabatic transfers of energy-momentum (i.e. infinitely slow processes in which $\delta p^\nu = \delta \mathcal{W}^\nu$) are reversible (see [24], section 6, for another example). Note also that the reversibility of this transformation has been justified using only condition (i), namely the second law of thermodynamics; no other property of the entropy has been invoked.

In order to show that this reversible process is a *pure* acceleration, which would prove van Kampen's postulate, see equation (3), we only need to show from micro-physics that necessarily

² Nothing forbids us to impose $\epsilon < 0$ in the action (7). In fact, changing the sign of ϵ keeping ϕ fixed is equivalent to keeping ϵ fixed and changing the sign of ϕ .

$$\delta M \sim \epsilon^2, \tag{15}$$

as this would immediately imply that $(\delta M)_{N \text{ kicks}} \sim 1/N \rightarrow 0$. The next two sections of the paper contain two alternative proofs of (15).

4 Variation of the Mass Induced by a Kick: Field Theory Approach

We derive equation (15) from a field theory point of view.

4.1 Classical Case

Let us define the tensor field

$$T^\mu_\nu = \mathcal{L}_{\text{Body}} \delta^\mu_\nu - \frac{\partial \mathcal{L}_{\text{Body}}}{\partial(\partial_\mu \varphi_i)} \partial_\nu \varphi_i, \tag{16}$$

where we are applying Einstein’s summation convention also to the label i . Given that the Euler-Lagrange equations, computed from the action (7), are

$$\frac{\partial \mathcal{L}_{\text{Body}}}{\partial \varphi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\text{Body}}}{\partial(\partial_\mu \varphi_i)} \right) = -\epsilon \phi \frac{\partial G}{\partial \varphi_i}, \tag{17}$$

one can easily show that T^μ_ν obeys the equation

$$\partial_\mu T^\mu_\nu = -\epsilon \phi \partial_\nu G. \tag{18}$$

This implies that for $t \leq 0$ and $t \geq \tau$, i.e. in those space-time regions in which $\phi = 0$, the tensor field T^μ_ν is conserved, namely $\partial_\mu T^\mu_\nu = 0$. Indeed, T^μ_ν is the Noether stress energy tensor associated with $\mathcal{L}_{\text{Body}}$ [17], therefore it can be used to define the four-momentum of the body before and after the kick, by means of the formulas

$$\begin{aligned} p_\nu &= \int T^0_\nu d^3x && \text{for } t \leq 0 \text{ (before the kick)} \\ p_\nu + \delta p_\nu &= \int T^0_\nu d^3x && \text{for } t \geq \tau \text{ (after the kick)}. \end{aligned} \tag{19}$$

Recalling that the four-velocity of the center of mass is $u^\nu = p^\nu/M$ and applying Gauss’ theorem to the spacetime region $\mathcal{R} = (0, \tau) \times \mathbb{R}^3$ (assuming that the body is finite, so that the fields are zero at infinity), one can use (18) to prove that

$$\delta M = -u^\nu \delta p_\nu = \epsilon \int_{\mathcal{R}} \phi u^\nu \partial_\nu G d^4x. \tag{20}$$

The second equality in equation (20) is exact, whereas the first is valid up to the first order in ϵ . In the limit of small ϵ , we may use linear response theory and model G as the sum

$$G = G_0 + \epsilon G_1, \tag{21}$$

where G_0 is the value that the observable $G(\varphi_i)$ would have (on the spacetime point under consideration) if no kick were impressed on the body, while ϵG_1 describes the perturbation to G due to the kick. Let us focus on the function $G_0(x^\nu)$. If no kick were impressed on the body, the body would remain in a state of thermodynamic equilibrium, and would be drifting rigidly with constant four-velocity u^ν without experiencing any macroscopic deformation, because it would keep the equilibrium shape. This implies that statistically (i.e. once we average over the microscopic fluctuations) we must have

$$u^\nu \partial_\nu G_0 = 0. \tag{22}$$

This formula can be justified with the qualitative argument above, but it can also be proved rigorously from condition (ii), see Appendix B. If we plug (21) into (20), we obtain

$$\delta M = -u^\nu \delta p_\nu = \epsilon^2 \int_{\mathcal{R}} \phi u^\nu \partial_\nu G_1 d^4x \sim \epsilon^2, \tag{23}$$

which is what we wanted to prove (see equation (15) and recall that $\epsilon^2 = 1/N^2$). In conclusion, van Kampen’s postulate is valid, the entropy is Lorentz invariant and thermodynamics admits a covariant generalization.

4.2 Quantum Case

The above calculations are essentially the same if we move to a quantum context. Equation (18) becomes an operatorial identity (in the Heisenberg picture), while (20) becomes a Kubo formula for the quantum statistical average $-u_\nu \langle \hat{p}^\nu \rangle$. Equation (22) remains valid, if we interpret G_0 as the quantum statistical average $\langle \hat{G} \rangle_{\text{eq}}$, see Appendix B. No further assumption about the equilibrium density matrix needs to be invoked in the proof. For example, we do not need to assume it to be of Gibbs-like form [21], because this might point towards a von Neumann interpretation of the entropy, leading us back to circularity issues.

As a final comment, we remark that the Unruh effect [25] disappears in the limit in which the accelerations are adiabatic. In fact, with a simple order of magnitude estimate (see Appendix C.1), one can verify that

$$(\text{Unruh corrections}) \sim \frac{e^{-1/\epsilon}}{\epsilon}. \tag{24}$$

This shows that the Unruh effect is non-perturbative in ϵ : it decays to zero faster than any finite power of ϵ .

5 Variation of the Mass Induced by a Kick: Quantum Mechanics Approach

The proof of (15) given above, using a field theory approach, makes the role of condition (ii) manifest. However, it somehow hides the physical meaning of our result. Why does a small kick conserve (to the first order) the mass of a system of particles in equilibrium, while accelerating it? Why must it be that

$$\delta(p^\nu p_\nu) \sim \epsilon^2 \quad \text{whereas} \quad \delta p^\nu \sim \epsilon ? \tag{25}$$

In this section, we will show, with a simple quantum mechanical argument, that (25) is a consequence of the mathematical structure of the Poincaré group. The argument is rigorously formulated within relativistic quantum mechanics [26], while the connection with quantum field theory is somehow heuristic. This makes the argument that follows probably less conclusive than the one outlined in the previous section, but it gives a deeper insight into the dynamical origin of (25).

5.1 The Mass Spectrum of a Finite Body

For the total four-momentum p^ν to be finite, the body must be of finite size. But a completely isolated finite body in thermodynamic equilibrium must be self-bounded [22], otherwise it would eventually break up into smaller pieces in relative motion. It is well-known from ordinary quantum mechanics that bound states of many particles have a discrete mass spectrum (as we see, for example, in nuclear and atomic physics). The intuition behind this fact is that the degrees of freedom of a many-body system decouple into center of mass degrees of freedom plus internal degrees of freedom. Since, in a bound state, the particles cannot escape the conglomerate,³ the internal degrees of freedom (which describe essentially the relative positions between the particles) are bounded and, hence, have discrete energy eigenvalues. Recalling that the rest mass is the energy measured in the rest frame (i.e. it is the Hamiltonian of the internal degrees of freedom, see [26]), the discreteness of the mass eigenvalues follows.

Let us see the mathematical implications of the argument above. Given that the space-time translation operators \hat{p}^ν , computed from the Lagrangian density $\mathcal{L}_{\text{Body}}$, commute with each other, we can take, as basis of the Hilbert space of the body, some states

$$|p^\nu, a\rangle, \tag{26}$$

satisfying the eigenvalue equations

$$\hat{p}^\nu |p^\nu, a\rangle = p^\nu |p^\nu, a\rangle. \tag{27}$$

³ For large objects, at $T \neq 0$, perfect confinement is almost impossible and some form of radiation is always emitted. For this reason, relativistic thermodynamics is an idealization, which becomes valid in the limit in which the surface of bodies plays the role of a perfect mirror, keeping all the energy inside.

The additional quantum number a is arbitrary (it is used to break possible degeneracies) and can be taken discrete. The eigenvalues p^ν must be continuous (they organise themselves into three-dimensional hyperboloids), due to the mathematical structure of the Poincaré group [17]. The square mass operator

$$\hat{M}^2 := -\hat{p}^\nu \hat{p}_\nu \tag{28}$$

commutes with all the generators of the Poincaré group (computed from $\mathcal{L}_{\text{Body}}$) and is diagonal on the basis (26), with eigenvalue equation

$$\hat{M}^2 |p^\nu, a\rangle = m^2 |p^\nu, a\rangle \quad m^2 = -p^\nu p_\nu. \tag{29}$$

The scalar $m > 0$ can be interpreted as the mass of the state $|p^\nu, a\rangle$. Combining the fact that p^ν is “3D-continuous”, with the fact that m and a are discrete, we can conclude that

$$\langle \tilde{p}^\nu, \tilde{a} | p^\nu, a \rangle = 2p^0 (2\pi)^3 \delta^{(3)}(\tilde{p}^j - p^j) \delta_{\tilde{m}m} \delta_{\tilde{a}a}. \tag{30}$$

The standard normalization factor $2p^0 (2\pi)^3$ guarantees that (30) is Lorentz-invariant [19].

Equation (30) is crucial for us, because it shows that we can build a *normalisable* (i.e. physical) state $|\Psi\rangle$ which is eigenvector of the mass operator, namely

$$\hat{M}|\Psi\rangle = m|\Psi\rangle. \tag{31}$$

However, the same is not true for the individual components \hat{p}^ν : the physical state $|\Psi\rangle$ must be a wavepacket, namely a continuous superposition of eigenstates of \hat{p}^ν . As we are going to show, this is the central difference between $\hat{p}^\nu \hat{p}_\nu$ and \hat{p}^ν , which is responsible for the different scalings of the corresponding perturbations.

5.2 Kicking Mass Eigenstates

Due to the presence of the term $\epsilon \phi G$ in the action (7), the operators \hat{p}^ν (which are computed from $\mathcal{L}_{\text{Body}}$) are not conserved during the kick. As a first step, let us compute the variation of the mass of the body, induced by a kick, when the initial state $|\Psi\rangle$ is an eigenvector of \hat{M} , satisfying the eigenvalue equation (31).

As $\phi(x^\nu)$ is an assigned function of the coordinates, the evolution of the body is unitary (the final state is still a pure state); this is the definition of thermal isolation [23] or, equivalently, of no heat transfer [3]. Working in the Schrödinger picture, we may call $|\Psi_\epsilon(\tau)\rangle$ the state of the body at the time τ (just after the perturbation has been switched off) as a function of the coupling constant ϵ , parameterizing the intensity of the kick. Clearly, for $\epsilon = 0$, the mass is conserved (no kick has occurred), so that we may write

$$\delta M(\epsilon) = \frac{\langle \Psi_\epsilon(\tau) | \hat{M} | \Psi_\epsilon(\tau) \rangle}{\langle \Psi_\epsilon(\tau) | \Psi_\epsilon(\tau) \rangle} - \frac{\langle \Psi_0(\tau) | \hat{M} | \Psi_0(\tau) \rangle}{\langle \Psi_0(\tau) | \Psi_0(\tau) \rangle}. \tag{32}$$

Expanding this function to the first order in ϵ we obtain

$$\delta M(\epsilon) = \epsilon \frac{d}{d\epsilon} \left(\frac{\langle \Psi_\epsilon(\tau) | \hat{M} | \Psi_\epsilon(\tau) \rangle}{\langle \Psi_\epsilon(\tau) | \Psi_\epsilon(\tau) \rangle} \right) \Big|_{\epsilon=0} + \mathcal{O}(\epsilon^2). \tag{33}$$

If we compute the derivative in ϵ explicitly, we get

$$\delta M(\epsilon) = \epsilon \frac{\langle \Psi' | \Delta \rangle + \langle \Delta | \Psi' \rangle}{\langle \Psi_0(\tau) | \Psi_0(\tau) \rangle} + \mathcal{O}(\epsilon^2), \tag{34}$$

with

$$\begin{aligned} |\Psi'\rangle &= \frac{d|\Psi_\epsilon(\tau)\rangle}{d\epsilon} \Big|_{\epsilon=0} \\ |\Delta\rangle &= \hat{M}|\Psi_0(\tau)\rangle - \frac{\langle \Psi_0(\tau) | \hat{M} | \Psi_0(\tau) \rangle}{\langle \Psi_0(\tau) | \Psi_0(\tau) \rangle} |\Psi_0(\tau)\rangle. \end{aligned} \tag{35}$$

The final step consists of realising that, if the initial state $|\Psi\rangle$ obeys equation (31), then

$$\hat{M}|\Psi_0(\tau)\rangle = m|\Psi_0(\tau)\rangle, \tag{36}$$

because, when $\epsilon = 0$, the Hamiltonian is \hat{p}^0 , which commutes with \hat{M} . Inserting (36) into the second equation of (35) we find $|\Delta\rangle = 0$, which immediately implies

$$\delta M(\epsilon) \sim \epsilon^2. \tag{37}$$

It is interesting to note that this result does not depend on the details of the full Hamiltonian of the system, because the explicit formula for $|\Psi'\rangle$ is completely irrelevant. However, the assumption that $|\Psi\rangle$ is a mass eigenstate is crucial. If we repeat the calculations above, taking as initial state a superposition

$$\frac{|m_1\rangle + |m_2\rangle}{\sqrt{2}}, \tag{38}$$

$|m_1\rangle$ and $|m_2\rangle$ being two normalised mass eingenstates, relative to two different eigenvalues m_1 and m_2 , we now obtain (truncating to the first order in ϵ)

$$|\Delta\rangle = \frac{m_1 - m_2}{2\sqrt{2}} \left(|m_1(\tau)\rangle - |m_2(\tau)\rangle \right) \tag{39}$$

which does not vanish. By analogy, it becomes immediately clear why, in a kick, one is always able to induce an acceleration: any physical state must be a superposition of eigenstates of \hat{p}^ν , hence the variation of $\langle \Psi | \hat{p}^\nu | \Psi \rangle$ is of order ϵ for the same reason why the variation of $(\langle m_1 | + \langle m_2 |) \hat{M} (|m_1\rangle + |m_2\rangle)$ is of order ϵ .

5.3 Kicking Thermal States

As we explained qualitatively in subsection 4.1 (and proved rigorously in Appendix B), a system that is in thermodynamic equilibrium has constant shape. Its internal structure is conserved over time and the only change that the system can experience is a rigid macroscopic motion. Given that \hat{M} is the Hamiltonian of the internal degrees of freedom, it immediately follows that the density matrix of a macroscopic body in equilibrium satisfies the equation

$$\left[\hat{\rho}_{\text{eq}}, \hat{M} \right] = 0. \tag{40}$$

It is not hard to show that this condition is essentially equivalent to equation (61) of Appendix B.⁴

Equation (40) implies that there is an orthonormal set of mass eigenstates $|\Psi^{(n)}\rangle$, with

$$\hat{M}|\Psi^{(n)}\rangle = m_n|\Psi^{(n)}\rangle \quad \langle \Psi^{(\tilde{n})} | \Psi^{(n)} \rangle = \delta_{\tilde{n}n}, \tag{41}$$

such that

$$\hat{\rho}_{\text{eq}} = \sum_n \mathcal{P}_n |\Psi^{(n)}\rangle \langle \Psi^{(n)}| \tag{42}$$

with

$$\mathcal{P}_n > 0 \quad \sum_n \mathcal{P}_n = 1. \tag{43}$$

Taking this as initial state and recalling that the evolution is unitary, it follows that the average value of \hat{M} at a time τ (at the end of the kick) is

$$M_\tau = \sum_n \mathcal{P}_n \frac{\langle \Psi_\epsilon^{(n)}(\tau) | \hat{M} | \Psi_\epsilon^{(n)}(\tau) \rangle}{\langle \Psi_\epsilon^{(n)}(\tau) | \Psi_\epsilon^{(n)}(\tau) \rangle}. \tag{44}$$

Given that equation (37) applies to each contribution in the sum over n (because each state $|\Psi_\epsilon^{(n)}(\tau)\rangle$ is the time-evolved of a mass eigenstate), it applies also to a body with density matrix $\hat{\rho}_{\text{eq}}$, completing our proof.

There is a final remark that we need to make. All our analysis was performed within the assumption that the system does not radiate particles as a result of the kick (particles can be created and destroyed *inside the body*, but no particle can abandon the body). This is an important assumption, because, if it happens that the system emits particles along the way, the calculations above remain valid, but the quantity M can no longer be interpreted as the mass of the body alone, but as the

⁴ There is a small difference between (40) and (61), which is due to the fact that the present description is entirely quantum-based, while in the appendix we use a hybrid approach. For macroscopic bodies this difference becomes negligible and the final result is the same.

rest-frame energy of the *total system* (“body + emitted particles”) invalidating the assumptions that lead to (12). Luckily, one can easily prove (see Appendix C.2) that also the probability of stimulated emissions induced by a kick is of the order ϵ^2 (and, therefore, vanishes for adiabatic accelerations).

6 Conclusions

We have proved that the equation of state of isolated moving bodies (including only the four-momentum among the relevant variables) is always $S = S(M)$. Rather than showing this by arbitrarily postulating the Lorentz covariance of the laws of thermodynamics, we have focused on the dynamical consequence of assuming $S = S(M)$. In fact, declaring that two macroscopic states ψ and ψ' have the same entropy is equivalent to stating that there must be an adiabatic transformation that leads from ψ to ψ' and vice-versa. Using tools from both classical and quantum field theory we have shown that, indeed, infinitely slow accelerations, generated by a time-dependence of the Hamiltonian, must conserve the rest mass of bodies initially in thermodynamic equilibrium, making $S = S(M)$ the only equation of state possible.

This sets solid foundations for relativistic thermodynamics and, again, shows that the axiomatization proposed by [15] and [16] is the only one possible. Furthermore, this paper complements our previous study on the nature of the temperature [22], in that it clarifies further the meaning of the work four-vector $\delta\mathcal{W}^\nu$. In the same way in which one may intuitively decompose the heat four-vector $\delta\mathcal{Q}^\nu$ into time and space components as

$$\delta\mathcal{Q}^\nu = \begin{pmatrix} \text{Heat} \\ \text{Friction} \end{pmatrix}, \tag{45}$$

one may consider the analogous (non-rigorous but useful) pictorial decomposition of the work four-vector as

$$\delta\mathcal{W}^\nu = \begin{pmatrix} \text{Work} \\ \text{Kick} \end{pmatrix}. \tag{46}$$

In the same way in which a wall can exert work on a gas (changing its energy), a potential can induce a kick on a freely moving body (changing its momentum). Both these processes, if executed slowly enough, become reversible. The first becomes the standard pressure-volume (“PdV”) adiabatic work, while the second becomes a pure acceleration.

Appendix

Planck's Original Argument Revisited

Here we present a new version of Planck's original argument [14], which is slightly less abstract, but logically equivalent.

Let S_X be the entropy of a body X , as measured in its own rest frame, and assume that, if an observer A (say, Alice) sees X moving with a given speed v_{XA} , she will attribute to X an entropy

$$S_A = \gamma_{XA}^n S_X \quad \gamma_{XA} = (1 - v_{XA}^2)^{-1/2}, \quad (47)$$

with n a constant exponent to be determined. This is a reasonably general assumption about the transformation law of the entropy, as it includes the possibility for the entropy to be a scalar ($n = 0$), the zeroth component of a four-vector ($n = 1$) or an arbitrary power of it.

Let us consider a second observer B (say, Bob), in motion with respect to Alice. Bob should assign an entropy to X using a rule that is analogous to that of Alice (there is nothing special about Alice's frame), namely

$$S_B = \gamma_{XB}^n S_X \quad \gamma_{XB} = (1 - v_{XB}^2)^{-1/2}, \quad (48)$$

where v_{XB} is the speed of X with respect to Bob. Now, assume that X is initially at rest with respect to Alice (namely, $\gamma_{XA} = 1$) and consider an infinitesimal reversible transformation in which X is slowly set into motion. If the transformation is reversible, it should conserve the entropy in Alice's reference frame, hence

$$0 = \delta S_A = \delta(\gamma_{XA}^n S_X) = \gamma_{XA}^n \delta S_X = \delta S_X, \quad (49)$$

where we have used the fact that

$$\delta \gamma_{XA} = \gamma_{XA}^3 v_{XA} \delta v_{XA} = 0, \quad (50)$$

because initially $v_{XA} = 0$. On the other hand, it should be reversible also in the reference frame of Bob, because reversibility is a statement about the possibility of both the process itself and its inverse to occur, which cannot depend on the observer. Hence, using (49), we obtain

$$0 = \delta S_B = \delta(\gamma_{XB}^n S_X) = n S_X \gamma_{XB}^{n+2} v_{XB} \delta v_{XB}. \quad (51)$$

Considering that in general $v_{XB} \delta v_{XB} \neq 0$, the exponent n must vanish and, consequently, the entropy must be a scalar.

Isolated Bodies in Equilibrium Move Rigidly

Here we prove that, when a body in thermodynamic equilibrium is perfectly isolated (also dynamically, hence $\epsilon = 0$), it drifts rigidly at constant velocity without experiencing any change of shape. The key assumption to be used in the proof is condition (ii), see subsection 2.2. We recall that all the calculation are performed, for clarity, in the preferred reference frame A introduced in condition (i).

Centroids

It is always possible to build, starting from the Noether stress-energy tensor $T^{\nu\rho}$, which is not necessarily symmetric, the Belinfante-Rosenfeld stress-energy tensor $\Theta^{\nu\rho}$, which is symmetric [17]. This tensor field can be used to write the angular momentum tensor $J^{\nu\rho}$, which is the generator of the Lorentz group and is conserved (for isolated bodies), in the form

$$J^{\nu\rho} = \int (x^\nu \Theta^{\rho 0} - x^\rho \Theta^{\nu 0}) d^3x. \tag{52}$$

This formula can be used to show that the conservation of J^{0j} implies [27]

$$\frac{dx_{CD}^j}{dt} = \frac{p^j}{p^0} = \frac{u^j}{u^0} =: v^j, \tag{53}$$

where x_{CD}^j is the position of the centroid of the system, defined as

$$x_{CD}^j = \frac{1}{p^0} \int x^j \Theta^{00} d^3x. \tag{54}$$

This well-known fact is the relativistic generalization of the Newtonian law according to which the center of mass evolves following a uniform rectilinear motion. No matter how complicated the internal dynamics of the body is, as long as the body is isolated, condition (ii) guarantees that

$$x_{CD}^j(t) = x_{CD}^j(0) + v^j t. \tag{55}$$

The Density Matrix at Equilibrium

Consider a body in equilibrium with four-momentum p^ν and centroid x_{CD}^j , at a given time. The state of the system must be a function of these parameters. Since in quantum mechanics any physical state can be modelled through a density matrix, there must be a formula

$$\hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j], \tag{56}$$

which gives all the physical properties of a system in thermodynamic equilibrium as a function of its four-momentum p^ν and of its centroid x_{CD}^j . No other free parameter needs to be included in (56) because we are assuming that there are no (relevant) additional constants of motion. We do not specify any precise formula for (56), because this would imply giving a statistical interpretation to the entropy, which is something we want to avoid here.

If \hat{p}^ν are the four-momentum operators, then the unitary operator

$$\hat{U}(\Delta x^\nu) = \exp(-i\hat{p}_\nu \Delta x^\nu) \tag{57}$$

is a space-time translation, which acts on the field operators $\hat{\phi}_i$ as follows [17]:

$$\hat{U}(\Delta x^\nu) \hat{\phi}_i(x^\nu) \hat{U}^\dagger(\Delta x^\nu) = \hat{\phi}_i(x^\nu + \Delta x^\nu). \tag{58}$$

Clearly, if we consider a system in thermodynamic equilibrium, and we operate on it a pure translation in space, the final state must still be an equilibrium state, namely

$$\hat{U}(\Delta x^j) \hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j] \hat{U}^\dagger(\Delta x^j) = \hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j + \Delta x^j]. \tag{59}$$

Rigid Motion

By definition, if an isolated body is in equilibrium at a given time, it is also in equilibrium at later times. Hence, recalling that the four-momentum is conserved and that equation (55) must hold, we have

$$e^{-i\hat{p}^0 \Delta t} \hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j] e^{i\hat{p}^0 \Delta t} = \hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j + v^j \Delta t]. \tag{60}$$

Using equation (59), we find

$$\hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j] = \hat{U}\left(\frac{u^\nu \Delta t}{u^0}\right) \hat{\rho}_{\text{eq}}[p^\nu, x_{CD}^j] \hat{U}^\dagger\left(\frac{u^\nu \Delta t}{u^0}\right). \tag{61}$$

Multiplying this equation by an arbitrary field of observables $\hat{G}(x^\nu)$ and taking the trace of the result, we find

$$\text{tr}\left[\hat{\rho}_{\text{eq}} \hat{G}(x^\nu)\right] = \text{tr}\left[\hat{\rho}_{\text{eq}} \hat{G}\left(x^\nu - \frac{u^\nu \Delta t}{u^0}\right)\right]. \tag{62}$$

Taking the derivative of this formula with respect to Δt we finally obtain

$$u^\nu \partial_\nu \langle \hat{G} \rangle_{\text{eq}} = 0, \tag{63}$$

which is what we wanted to prove.

Note that this result is in perfect agreement with the relativistic formulation of the zeroth law of thermodynamics. In fact, following [22], in thermal equilibrium there must be one reference frame in which perfect stationarity is achieved. From (63), we

see that this reference frame is identified by the four-velocity $u^\nu = p^\nu/M$, in agreement with [22].

Effect of Absorption/Emission Processes on Adiabatic Accelerations

In C.1 we show that the Unruh effect does not play any role in adiabatic accelerations. In C.2 we show that the probability of emission of particles out of the body, stimulated by the kick, is of order ϵ^2 .

Unruh Effect for Adiabatic Accelerations

From the point of view of a particle detector that accelerates with constant acceleration a , the average number of particles (scalar bosons, for simplicity) with energy E is given by [28, 29]

$$f(E) = \frac{1}{e^{2\pi E/a} - 1}. \tag{64}$$

The detector can make level transitions by absorbing and emitting particles, with a stimulated absorption/emission rate which is proportional to $f(\Delta m)$, where $\Delta m > 0$ is the mass separation between two levels of the detector. Therefore, if a detector experiences a uniform acceleration a , for an interval of proper time t , the transition probabilities (associated with Unruh-particle absorption/emission processes) scale as [30]

$$\mathcal{P}_{Unruh} \sim \frac{t}{e^{2\pi \Delta m/a} - 1}. \tag{65}$$

For a reversible acceleration, as described in subsection (3.2), $t \sim N$, while $a \sim 1/N$, hence

$$\mathcal{P}_{Unruh} \sim \frac{N}{e^N - 1} \sim Ne^{-N} \longrightarrow 0. \tag{66}$$

Therefore, in the limit of adiabatic accelerations ($N \rightarrow +\infty$), the corrections due to the Unruh effect are exponentially suppressed.

Can an Adiabatic Acceleration Stimulate Emissions?

The Hilbert space of the body, generated by the discrete-mass basis $|p^\nu, a\rangle$, introduced in equation (26), is only a subspace of the full Hilbert space upon which the quantum fields $\hat{\varphi}_i$ act (as operators). There are many other states, including, in particular, states in which the body coexists with other particles. These states constitute the continuous part of the mass spectrum [19]. The projector

$$\hat{P} := \hat{\mathbb{I}} - \sum_{m,a} \int \frac{d^3p}{(2\pi)^3} \frac{|p^\nu, a\rangle \langle p^\nu, a|}{2p^0} \quad (67)$$

projects onto this second part of the Hilbert space ($\hat{\mathbb{I}}$ is the identity operator acting on total Hilbert space of the field theory). Given a normalised state $|\Psi\rangle$, the average $\langle \Psi | \hat{P} | \Psi \rangle$ is the probability that we observe “something that is not just the body alone”. Therefore, we can interpret the quantum average

$$\mathcal{P}_{\text{Em}}(\tau) = \text{tr} \left[\hat{\rho}(\tau) \hat{P} \right] \quad (68)$$

as the probability that the body has emitted something during a kick. Now, from equation (27) it follows that (as long as $\epsilon = 0$)

$$\left[\hat{P}, \hat{p}^0 \right] = 0. \quad (69)$$

Furthermore, since in the initial state there is only the body, we know that

$$\hat{\rho}(0) \hat{P} = 0. \quad (70)$$

Combining (69) with (70) we immediately find that, if $\epsilon = 0$, $\mathcal{P}_{\text{Em}}(\tau)$ vanishes. On the other hand, $\mathcal{P}_{\text{Em}}(\tau) \geq 0$ (recall that $\hat{P} = \hat{P}^\dagger = \hat{P}^2$), therefore we cannot impose $\mathcal{P}_{\text{Em}}(\tau) \sim \epsilon$, because ϵ has arbitrary sign. Hence, the leading order must be

$$\mathcal{P}_{\text{Em}}(\tau) \sim \epsilon^2, \quad (71)$$

or higher (but even), which is what we wanted to prove.

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Chapter 4

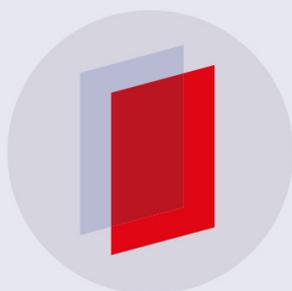
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PAPER

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Thermodynamics of uncharged relativistic multifluids

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Abstract

The internal layers of neutron stars are expected to contain several superfluid components that can significantly affect their dynamics. The description of such objects should rely on hydrodynamic models in which it is possible to unambiguously assign the value of the thermodynamic variables from microscopic calculations of the properties of matter. In this work we consider the phenomenological approach to multifluids modelling championed by Carter and, studying the relaxation of the system towards equilibrium, we assign a precise thermodynamic interpretation to its variables. We show that in thermodynamic equilibrium the equation of state contains less state variables than those needed in the phenomenological model, implying the existence of a gauge freedom of the theory that can be used to simplify the hydrodynamic formulation in the non-dissipative limit. Once this is understood, it becomes easy to translate the different multifluid formalisms that have been proposed in the literature into Carter's form. Finally, we show that the usual concepts of affinity and reaction coordinates, as they are introduced in chemistry, are not affected by the presence of superfluid currents. In an effort to make the concepts clear, the formalism is developed step-by-step from first principles, providing model examples and several applications of practical relevance for the study of superfluid neutron star interiors.

Keywords: fluid dynamics, neutron star, superfluidity

1. Introduction

Superfluidity in neutron star interiors opens the door to a wide range of exotic scenarios where metastable states in which two or more persistent currents can flow with respect to each other can be sustained for an extremely long time [1–3]. Therefore, we are forced to extend in a relativistic context the notion of fluid to include non-divisible substances which are crossed

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by more than one independent currents, namely a multifluid. The dynamics of multifluids is intrinsically different from that of ordinary mixtures, since in the former case the fluid components may have a different four-velocity also in the non-dissipative limit [4]. Apart from a richer dynamics, this fundamental difference introduces new properties also at the thermodynamic level since equilibrium states supporting persistent currents are possible, and observed, in terrestrial superfluid systems [5].

The Lagrangian approach to multifluid hydrodynamics championed by Carter and coworkers provides a powerful covariant formalism to construct phenomenological multifluid models [6]. However, our ability to reproduce the dynamics of a realistic system rests ultimately on the equation of state, which is usually a product of some microscopic calculations that are carried out in a particular frame. Since multifluid models in general relativity must be considered for an accurate modelling of superfluid neutron star interiors, the problem of how to embed the information given by a microphysical equation of state within a covariant phenomenological model is of primary importance: the wide range of applications spans from equilibrium models of stellar structure, see e.g. [7, 8], to the study of stellar oscillations, see e.g. [9, 10], pulsar glitches in general relativity, e.g. [11, 12], and cooling models [13].

Leaving aside the major theoretical concern about how to include in a covariant formalism dissipative effects (see e.g. [14] for an approach to the issue within the multi-fluid paradigm) and elastic components [15, 16] needed to describe the crust of neutron stars [17], also the non-dissipative limit of the theory may challenge the current understanding when applied to systems containing several species. In fact, we find that a formalism based on a master function allows for a gauge freedom when several non-superfluid species are present: the sophistication of the action principle, seen as the machinery which encodes the dynamics of the multifluid in the adiabatic limit, gives rise to a separation between the physical quantities and some redundancies which are required at the mathematical level. Our aim is to provide a physically motivated thermodynamic language that is gauge invariant.

The multifluid thermodynamics developed in the present paper is a useful tool not only for the description of perfect multifluids, but also to discuss some physical situations in which dissipation occurs. As a first application, we employ the gauge freedom to revisit the hydrodynamical model of [18], which encodes a dissipative force associated with chemical reactions in a multifluid. In real neutron stars this additional dissipative mechanism is related to beta decays and interactions between protons, neutrons and electrons and it is sometimes referred to as ‘rocket effect’ [19], in analogy with the evolution of a rocket whose mass is changing as it consumes its fuel.

The paper is organized as follows.

In section 2 we recall the variational approach of [20], see also [4], to set the notation and introduce a general phenomenological model for the dynamics of a perfect (i.e. non-dissipative) multifluid.

Section 3 is devoted to deriving all the equilibrium properties of a multifluid in contact with a heat bath, using only the expression of the energy–momentum tensor derived in section 2, the second law of thermodynamics and the irrotationality of the superfluid momentum.

In section 4 we use the tetrad formalism to recast the equations of non-dissipative multifluid hydrodynamics in a way that only variables with well defined thermodynamic interpretation appear; this reveals the connection between the Lagrangian approach and the one developed by [21] and later generalized by [10].

In section 5 we give an example of how it is possible to employ our thermodynamic approach to model dissipative phenomena, studying the problem of the relaxation to chemical equilibrium in a generic multifluid and showing that the two formulations of [18] and [22] are

equivalent. We analyze how the so-called ‘rocket-effect’ changes under a gauge transformation and use this result to reduce its form to the one proposed by [19] in the low temperature limit.

The appendices are devoted to discussing of some issues of practical relevance for neutron star modelling. In particular, in appendix C we solve a dictionary problem between two different formulations for the hydrodynamics in the outer core of a neutron star developed by [23] and [24].

Throughout the paper we adopt the spacetime signature $(-, +, +, +)$, choose units with the speed of light $c = 1$ and Newton’s constant $G = 1$, use greek letters $\nu, \rho, \sigma \dots$ for coordinate tensor indices.

2. Relativistic multifluid model

In this section, we briefly review the variational principle for non-dissipative multifluid hydrodynamics introduced in [20] and [25]. This allows us to set up notation and to remark some subtleties (like the distinction between the Lagrangian and the master function for the system) which will be important in the following sections.

2.1. The variational approach for non-dissipative models

The fundamental ingredients in a multifluid theory are the currents, one for each of the chemical species comprising the system. Those currents are vector fields whose norm represents the local density of the relative chemical species² (measured in the frame in which the species is at rest) and whose direction is collinear to its average four-velocity. An additional ‘thermal current’ s^ν carries information about the local frame, identified by the four-velocity u^ν say, in which the excitations present in the system thermalize (i.e. have average zero velocity) and about the entropy of the system. Hence, it is always possible to write the thermal current as

$$s^\nu = s u^\nu, \quad (1)$$

where the scalar $s = \sqrt{-s_\mu s^\mu}$ is the entropy per unit volume measured in the local aforementioned frame.

For reasons which will become clear in the following section we assume that the remaining independent currents can be divided into two subsets. We suppose that a number l of them are forced to comove with the entropy, so we call them ‘ s -locked’ and count them with the labels A, B running from 0 to l (0 identifies the thermal current). According to this notation, the s -locked currents can be written as

$$n_A^\nu = n_A u^\nu. \quad (2)$$

Since there is no net entropy flow in the frame identified by u^μ , no heat transport is measured by an observer comoving with the s -locked components.

We assume the remaining k currents to be completely independent; the labels i, h (running from 1 to k) are used to count them. Since they can flow in an arbitrary direction we call them ‘free’ and in general we have

²In the present context the term ‘chemical species’ denotes a subset of the particles comprising the substance, which are grouped according to a non-specified criterion. From a general point of view, it is not necessary that all the particles present in the system contribute to the currents (e.g. the electrons in [18]), nor a multiple counting of the same particle is forbidden (e.g. the density of ions in [26]). However, the full set of currents (entropy included) should give without redundancies the complete information about the local state of matter. This is related to the ‘chemical gauge’ problem discussed in [26], see appendix B.

$$n_i^\nu = n_i u_i^\nu, \quad (3)$$

where u_i^ν can be different from u^ν . No Einstein summation convention will be used for species labels A, B, i, h .

Since we have a total number of $1 + l + k$ currents, there are $4(1 + k + l)$ variables that may be used to construct a Lagrangian for the system. However, only $l + 4(1 + k)$ of such variables are independent, namely s^ν , n_i^ν and the scalars n_A . Despite the fact that these are all the degrees of freedom required to define completely the local state of the multifluid at a point, the fields upon which the action will depend do not need to be so numerous.

The equations of motion in a spacetime region \mathcal{M} have to be found by imposing the stationarity of the total action I with respect to a variation of the fundamental fields of the theory [6]. As usual, I is the sum of a matter term I_m and of the Einstein–Hilbert action

$$I_{\text{EH}} = \int_{\mathcal{M}} \frac{R}{16\pi} \sqrt{-g} d_4x, \quad (4)$$

where R is the scalar curvature. A boundary term, unessential in the present discussion, is also necessary to give a well posed action principle (see e.g. [4]). In this way the Euler–Lagrange equations associated to a variation of the components of the metric are

$$G_{\nu\rho} = 8\pi T_{\nu\rho}, \quad (5)$$

where $G_{\nu\rho}$ is the usual Einstein tensor and

$$T_{\nu\rho} = -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g^{\nu\rho}}. \quad (6)$$

Since we are specializing our study to a non-dissipative and non-transfusive model, we follow the variational procedure proposed by [20] to ensure the conservation of the number density currents. In particular, the conservation laws

$$\nabla_\nu n_x^\nu = 0, \quad (7)$$

where the label x runs over all the currents (entropy included), should be automatically satisfied both on shell (i.e. on the solutions of the equations of motion) and off shell (i.e. on a generic spacetime-matter configuration which belongs to the domain of the action). This is done by realizing that the number of particles of type x contained in an infinitesimal volume element transported by the four-velocity field u_x^ν is conserved: all the information about the current is contained, once the initial condition is assigned, into three scalar fields giving the comoving (Lagrangian) coordinates of the volume element occupying the point in which they are evaluated [4, 20]. Therefore, we impose that the action depends only on the three scalar fields X_s^α , for $\alpha = 1, 2, 3$, describing the worldline congruence associated to u^ν and three more scalar fields X_i^α for every free component. In the end, apart from the metric, the total action I is a functional of $3(1 + k)$ independent fields:

$$I = I_{\text{EH}}[g] + I_m[g, X_s^\alpha, X_i^\alpha]. \quad (8)$$

In this way the s -locked components comove with the entropy both on shell and off shell. It is worth anticipating that the fact that the s -locked components and the entropy comove also off shell is particularly important as it gives rise to a gauge freedom, discussed in section 4.4. In section 4.5 it is shown how this freedom is related to the behaviour of the multifluid out of local thermodynamic equilibrium.

2.2. The master function and the equations of motion

In the previous section we introduced the basis for the most general multifluid theory where all the components are uncharged and there are no dissipative phenomena. The Lagrangian \mathcal{L} for the multifluid is introduced by imposing that the matter action has the local form

$$I_m = \int_{\mathcal{M}} \mathcal{L} \sqrt{-g} d_4x. \quad (9)$$

A general multifluid theory can be obtained by taking every species of the theory, no matter if they are independent or not, and constructing the symmetric matrix $n_{xy}^2 := -n_x^\nu n_{y\nu}$. We consider only the upper triangle $x \leq y$ comprised of

$$z = \frac{(1+k+l)(2+k+l)}{2} \quad (10)$$

terms and introduce the *master function*

$$\Lambda : \mathbb{R}^z \longrightarrow \mathbb{R}, \quad (11)$$

which is just an auxiliary function (a sort of pre-Lagrangian). At this point, the Lagrangian density is obtained from Λ as

$$\mathcal{L}[g, X_s^\alpha, X_i^\alpha] = \Lambda(-g_{\rho\nu} n_x^\rho [g, X_s^\alpha, X_i^\alpha] n_y^\nu [g, X_s^\alpha, X_i^\alpha]). \quad (12)$$

We make this distinction between \mathcal{L} and Λ , which is usually not found in literature, to remark that what matters for the dynamics is the value of \mathcal{L} on shell and off shell. On the other hand, Λ is an intermediate function that can receive arguments which are incompatible with the constraint of comoving s -locked components.

We assume Λ to be differentiable, which means that we are ignoring the possibility of first order phase transitions. We define the *bulk* coefficients

$$\mathcal{B}^x := -2 \frac{\partial \Lambda}{\partial n_{xx}^2}, \quad (13)$$

where, following [27], we adopt the notation $\mathcal{B}^s = \mathcal{C}$ for the one associated to the entropy. Furthermore, the so-called *anomaly* coefficients can be cast into a symmetric matrix \mathcal{A}^{xy} , whose upper triangle is defined as

$$\mathcal{A}^{xy} := -\frac{\partial \Lambda}{\partial n_{xy}^2} \quad \text{for } x < y. \quad (14)$$

The coefficients \mathcal{A}^{xy} are responsible for the *entrainment* effect when non-comoving species are present in the system, see [28, 29]. The partial derivatives are here interpreted in a strict

sense: they are derivatives of the master function Λ with respect to its argument n_{xy}^2 . However, if Λ is seen as a function of all the components of the $1+l+k$ currents *independently* (i.e. regardless of the fact that the s -locked currents are forced to comove with the entropy), then it can be seen as a function of $4(1+l+k)$ variables through the formula

$$\Lambda(n_x^\nu) := \Lambda(-g_{\rho\nu} n_x^\rho n_y^\nu). \quad (15)$$

Thanks to this operation, the canonically conjugate momenta are defined as

$$\mu_\nu^x := \frac{\partial \Lambda}{\partial n_x^\nu} = \mathcal{B}^x n_{x\nu} + \sum_{y \neq x} \mathcal{A}^{xy} n_{y\nu}, \quad (16)$$

where the sum runs over both the upper and lower triangles. Again, we stick to Carter's notation and the momentum μ_ν^x related to the entropy current is denoted by Θ_ν .

Following a derivation that is completely analogous to the one delineated e.g. in [4], it is possible to prove that the energy–momentum tensor is given by

$$T^\nu_\rho = \Psi \delta^\nu_\rho + \sum_x n_x^\nu \mu_\rho^x, \quad (17)$$

where the generalized pressure³ Ψ is related to the master function via (see also equation (2.3) in [30])

$$\Psi = \Lambda - \sum_x n_x^\rho \mu_\rho^x. \quad (18)$$

Finally, the hydrodynamic equations, obtained as Euler–Lagrange equations for the fields X_i^α and X_s^α , are

$$\begin{aligned} n_i^\rho \partial_{[\rho} \mu_{\nu]}^i &= 0 \quad \text{for } i = 1, \dots, k \\ \sum_{A=0}^l n_A^\rho \partial_{[\rho} \mu_{\nu]}^A &= 0. \end{aligned} \quad (19)$$

Once $\nabla_\nu n_x^\nu = 0$ is satisfied, the above system of equations implies that $\nabla_\rho T^{\rho\nu} = 0$, an information which is already contained in the Einstein's equations: the Euler–Lagrange equations associated to the worldline congruence of the entropy is redundant and could be replaced by the conservation of the total energy–momentum tensor.

3. Equilibrium properties of homogeneous systems

Even though it derives from an elegant mathematical framework, the variational model of section 2 is phenomenological, in the sense that it captures the formal properties needed to describe a multifluid in general relativity, but has no connection with the relevant physical properties of matter [22]: to apply it to physical systems, we need to provide a precise thermodynamic interpretation for its quantities.

In this section we show how to derive a thermodynamic description which is consistent with the variational model. The following discussion defines, starting from first principles, the connection of the phenomenological theory with the microphysics of the system and provides a new thermodynamic interpretation of many results which in other treatments required the inclusion of dissipation in order to be concretely proved.

3.1. Equilibrium with a heat bath

In thermodynamics it is conventional to consider a homogeneous macroscopic portion of the given substance, small enough to neglect gravity. To implement invariance under translations

³The generalized pressure can be defined by considering a stationary flow trough a channel: it is the pressure exerted by the fluid on the channel walls. To see this it is sufficient to take an orthonormal basis e_a such that e_1 is normal to the wall: since $n_x^1 = 0$ (otherwise the x component would flow through the wall), we have $T_{11} = \Psi$.

of the equilibrium state, while working with finite extensive quantities, the fluid is enclosed into a cubic volume $V = L^3$ with periodic boundary conditions. This box is a test spacetime, whose topology is $\mathbb{R} \times \mathbb{T}^3$, equipped with a chart (t, x^1, x^2, x^3) and the Minkowskian metric $\eta = \text{diag}(-1, 1, 1, 1)$. The periodic boundary conditions read $\{x^j = 0\} \equiv \{x^j = L\}$ for $j = 1, 2, 3$. It is useful to assume that the box is in contact with an ideal heat bath of constant temperature Θ_H , at rest in the global frame defined by the aforementioned chart, i.e. an effectively infinite mass-energy reservoir characterized by the equation of state

$$E_H(S_H) = E_{H0} + \Theta_H S_H, \quad (20)$$

where E_H and S_H are respectively the energy and entropy of the bath, measured by an observer \mathcal{O} such that $u_{\mathcal{O}} = \partial_t$. Imagine now to put a sub-system M inside the box, with an arbitrary initial condition: after an equilibration process involving a complex dissipative evolution partially driven by the interaction with the bath, M will reach a state of thermodynamic equilibrium. In particular, the second principle of thermodynamics, together with the conservation of the total energy of M plus the bath, implies that M will reach the state which minimizes the quantity

$$F = E_M - \Theta_H S_M, \quad (21)$$

where E_M and S_M are the energy and the entropy of the subsystem as measured in the frame of \mathcal{O} . In section 3.4 we will show that the quantity F , when the multifluid is in thermal equilibrium with the heat bath, becomes the Helmholtz free energy of the multifluid. Out of equilibrium, however, one should not be tempted to give thermodynamic interpretations to F , because the thermodynamic potentials may not be uniquely defined.

Assuming that we can associate an energy–momentum tensor to the substance constituting the system and that the final equilibrium state is homogeneous, the energy of M can be identified as

$$E_M = VT^{00}, \quad (22)$$

where the superscript 0 refers to t . Similarly, the total entropy S_M is

$$S_M = Vs^0, \quad (23)$$

where, according to (1), s^0 is the density of entropy measured by \mathcal{O} .

Let us now specialize the analysis to the multifluid case. As a starting hypothesis, the system M is prepared in an arbitrary initial condition, namely all the currents can be considered to be free. Combining equations (17) and (18) we obtain

$$T^{00} = -\Lambda + \sum_x n_x^j \mu_j^x, \quad (24)$$

where j runs over the spatial indices and obeys the Einstein summation convention. Plugging these results inside (21) we obtain that the equilibrium state minimizes the function

$$\mathcal{F} = F/V = -\Lambda + \sum_x n_x^j \mu_j^x - \Theta_H s^0. \quad (25)$$

Explicitly, this minimization condition reads (notice the raised index of the zeroth components of the momenta)

$$\delta\mathcal{F} = \sum_x (\mu^{x0} \delta n_x^0 + n_x^j \delta \mu_j^x) - \Theta_H \delta s^0 = 0 \quad (26)$$

for any dynamically allowed infinitesimal variation.

Following [31], let us introduce the manifold of homogeneous macrostates \mathcal{Z} , namely the set of all the states that can be realized in the box assuming all the currents (entropy included) to be free. A point in \mathcal{Z} , i.e. a generic macrostate, is identified by specifying the values of the variables of state $(n_x^\nu)_{x \in C}$, where C is the set of all the currents. If Λ has non-degenerate Hessian matrix in the variables n_x^j , the map

$$(n_x^\nu)_{x \in C} \longmapsto (n_x^0, \mu_j^x)_{x \in C} \quad (27)$$

is a bijection. Therefore, $(n_x^0, \mu_j^x)_{x \in C}$ is an alternative choice of coordinates on \mathcal{Z} .

In the absence of quantities that are conserved during the equilibration process, the variations δn_x^0 and $\delta \mu_j^x$ in (26) are unconstrained and independent with respect to each other, leading to the conditions $\mu^{x0} = 0$ for $x \neq s$. This would give the empty space as a unique solution. However, the presence of constants of motion imposes some constraints on the macrostates that the system can explore and, therefore, on the allowed variations. Only by taking into account these constraints it is possible to have non-trivial equilibria: this is explored in the following two sections.

3.2. Equilibration in the non-transfusive limit

Unconstrained variations in (26) give trivial equilibria. On the other hand, non-trivial equilibria are obtained when we restrict the allowed variations on a submanifold in \mathcal{Z} .

As a first case, consider the non-transfusive theory, where the number of particles of each chemical species is conserved. In this case, we have to consider the subspace in \mathcal{Z} defined by the constraints

$$n_x^0 = \frac{N_x}{V} \quad x \neq s, \quad (28)$$

where N_x is the number of particles of the species x and its value is imposed in the initial conditions in which the system is prepared before equilibration.

Apart from the particle number conservation, the presence of superfluid components in the multifluid introduces other constants of motion, whose conservation is directly linked to the possibility to have persistent currents at equilibrium [5].

Let us suppose that a generic component x is superfluid. Following [27], in the absence of vortices we have to impose that⁴

$$\partial_{[\rho} \mu_{\nu]}^x = 0. \quad (29)$$

This irrotationality condition is interpreted as resulting from the requirement that

$$\mu_\nu^x = \frac{k_x}{2\pi} \partial_\nu \phi^x, \quad (30)$$

where ϕ^x is the phase of a scalar order parameter for the species x under consideration. Here k_x is equal to $2\pi\hbar$ when x is a Bose fluid, while $\pi\hbar$ in the case of a Fermi fluid. The key assumption (30) is best justified by its physical consequences, as discussed by [32]: it is indeed the covariant form of the Josephson relation for neutral superfluids [26, 33].

⁴The irrotationality condition is not contained into the phenomenological model of the previous section, where the quantities μ_ν^x have no assigned physical meaning. To enforce internal consistency, this physical identification with the gradient of a phase has to be done by looking at the equations of motion of the phenomenological model: the first equation in (19) tells us that an irrotationality requirement can be imposed on the quantities μ_ν^j (and not on other combinations different from (16)).

Take now equation (29) for, say, $\rho = 0$ and $\nu = 1$ and integrate it along x^1 :

$$\frac{d}{dt} \int_0^L \mu_1^x(t, x^1, 0, 0) dx^1 = \mu_0^x(t, L, 0, 0) - \mu_0^x(t, 0, 0, 0). \quad (31)$$

Periodic boundary conditions imply that the right-hand side vanishes, so that the quantity⁵

$$W_1^x := \frac{1}{k_x} \int_0^L \mu_1^x(t, x^1, 0, 0) dx^1 \quad (32)$$

must be conserved during the evolution of the system, provided that the interaction with the heat bath does not destroy the order parameter anywhere (i.e. there are no zeroes of the order parameter). This can be also shown with a simple topological argument. Considering (30), W_1^x is the number of windings of the phase ϕ^x across the torus along the direction 1. Since μ_j^x are smooth functions on the spacetime, also W_1^x has to vary continuously with time. However, the only continuous functions from \mathbb{R} to \mathbb{Z} are the constant ones, which implies that W_1^x is conserved during the equilibration process with the heat bath.

For homogeneous configurations, the conservation of the winding numbers (similarly one can define also W_2^x and W_3^x) implies the further constraints

$$\mu_j^x = \frac{k_x W_j^x}{L}. \quad (33)$$

Therefore, equation (28) tells us that we have to impose $\delta n_x^0 = 0$ for any x , while (33) forces us to set $\delta \mu_j^x = 0$ when x refers to a superfluid species. With these constraints the condition (26) becomes

$$s^j \delta \Theta_j + \sum_{x \in \mathcal{N}} n_x^j \delta \mu_j^x + (\Theta^0 - \Theta_H) \delta s^0 = 0, \quad (34)$$

where \mathcal{N} is the set of the normal (i.e. non-superfluid) species. Because of its special role in the relaxation process, we do not include the entropy in the set \mathcal{N} . Thus, the term $\Theta^0 \delta s^0 + s^j \delta \Theta_j$, which in equation (26) is included in the summation over x , has been made explicit here.

Now, the remaining infinitesimal variations appearing in the left-hand side of the above equation are free and independent with respect to each other, implying that $\delta \mathcal{F}$ is zero only if all the prefactors vanish. Let us examine the equations we get one by one. First of all, we have

$$s^j = 0 \quad j = 1, 2, 3. \quad (35)$$

This equation states that the excitations of the system thermalize in a way to be on average at rest in the frame of the heat bath. This can also be interpreted by saying that, in conditions of thermal equilibrium, the four-velocity u^ν , defined in (1), identifies the observer in whose reference frame the multifluid can be studied using the canonical ensemble.

Secondly, for the normal species we have

$$n_x^j = 0 \quad j = 1, 2, 3 \quad x \in \mathcal{N}. \quad (36)$$

Due to the absence of conserved quantities other than the particle numbers, a normal component initially flowing with respect to the heat bath will be slowed down by an entropic force arising from the interaction with the bath, until the flow stops. This, however, does not happen to the superfluid currents, which can survive at a thermodynamic level as a result of the

⁵This quantity could be defined by integrating the momentum along any path that is homotopy equivalent to a loop embracing the torus in the direction x^1 exactly once.

conservation of the winding numbers⁶ W_j^x . Equations (35) and (36) combined imply that thermodynamic equilibrium is reached when all the normal currents comove with the entropy (i.e. they are s -locked). Of course this is not necessarily true out of equilibrium, when dissipation occurs.

The last equation we get from (34) is

$$\Theta^0 = \Theta_H. \quad (37)$$

As expected, when the system is in equilibrium with the bath, its temperature must coincide with Θ_H ; together with (35), this statement assures that the absolute temperature Θ of a multifluid in thermodynamic equilibrium can be expressed in a covariant fashion as

$$\Theta = -\Theta_\nu u^\nu. \quad (38)$$

Clearly, this coincides with the value Θ^0 measured by an observer comoving with the s -locked components.

3.3. Equilibration in the presence of transfusion

In the previous subsection we have computed the equilibrium properties of a multifluid in the non-transfusive limit. Let us see what happens if we relax this assumption and assume that a reaction

$$\sum_{x \neq s} \alpha_x X_x \rightleftharpoons \sum_{x \neq s} \beta_x X_x \quad (39)$$

can occur and that no other ways to modify N_x are allowed. Here X_x is the name of the chemical species associated with the current x , while α_x and β_x are the stoichiometric coefficients. Let us introduce the *reaction coordinate* N , also called in chemistry *extent of reaction*, which counts the number of reactions from the left to the right minus the number of reactions from the right to the left, setting the zero on the initial condition. Starting from a state in which the

particle numbers are $N_x^{(in)}$, we have

$$N_x = N_x^{(in)} + (\beta_x - \alpha_x)N. \quad (40)$$

This means that the variations $\delta n_x^0 = \delta N_x/V$ are not forced to be zero any more, but they are still not completely free. Instead, they assume the form

$$\delta n_x^0 = (\beta_x - \alpha_x) \delta n^0, \quad x \neq s, \quad (41)$$

where we have defined⁷ $n^0 := N/V$. As discussed in the previous section, the conservation of the winding numbers W_j^x is not affected by the reactions. Hence, in this case equation (26) reduces to

⁶The superfluid momentum is modified by the motion of physical boundaries or of quantized vortices [34]. This may be seen as the physical manifestation of the conservation of the winding numbers in an annulus. It is also possible to give a more microscopic interpretation, at least for a bosonic superfluid [35]: as a result of the interaction with the bath, particles may be knocked out of the condensate; this process reduces the amplitude of the order parameter associated to condensation, but it will not alter its phase. Hence, changing the momentum of the superfluid would require a transition that involves a macroscopic number of particles from one state to another simultaneously, a very low probability process.

⁷For now, the index 0 of n^0 is a pure notation convention. In section 5.1 we give an interpretation of n^0 as the zeroth component of a virtual current, giving a formal justification for this convention.

$$s^j \delta \Theta_j + \sum_{x \in \mathcal{N}} n_x^j \delta \mu_j^x + (\Theta^0 - \Theta_H) \delta s^0 + \sum_{x \neq s} (\beta_x - \alpha_x) \mu^{x0} \delta n^0 = 0, \quad (42)$$

a form which is identical to (34) plus an additional term due to the presence of reactions.

The submanifold of the allowed macrostates is fully described by the variables $(n^0, s^0, \Theta_j, \mu_j^x)_{x \in \mathcal{N}}$, whose variations are independent. This means that equations (35)–(38) remain true, but we get one more condition arising from the presence of the additional term in (42),

$$\sum_{x \neq s} \alpha_x \mu^{x0} = \sum_{x \neq s} \beta_x \mu^{x0}. \quad (43)$$

Not surprisingly, this is the conventional formula for the chemical equilibrium. However, we obtained more than a well-known result: as (37) provides a first-principles justification of the notion of absolute temperature in (38), similarly this relation provides a physical interpretation of the notion of chemical potential. In fact, since (35) still holds, we define the thermodynamic chemical potentials (i.e. the quantities needed in the equation for chemical equilibrium) as

$$\mu_x^{(T)} := -\mu_\nu^x u^\nu. \quad (44)$$

In this way, the chemical equilibrium condition can be written in terms of covariant quantities as

$$\sum_{x \neq s} \alpha_x \mu_x^{(T)} = \sum_{x \neq s} \beta_x \mu_x^{(T)}. \quad (45)$$

This result is perfectly coherent with [18], where beta reactions in superfluid neutron stars are taken into account.

As a final remark, notice that in the above analysis we have treated N as a completely free variable, without considering the fact that N_x must be non-negative in (40). The variable N should move along a segment whose extrema are reached when we run out of a reactant and it can happen that the minimum of the free energy is reached on the border of this segment, for example when one reaction always dominates over the other and does not stop until all its reactants are transformed into products. In this case equation (45) needs not to be satisfied.

3.4. Thermodynamic potentials

The thermodynamic properties of a system are completely known once an equation of state is given. Since all the theory describing a multifluid is determined once a master function Λ is assigned, it is usually stated that

$$\Lambda = \Lambda(n_{xy}^2) \quad (46)$$

represents the equation of state of the multifluid. Even if Λ contains all the information needed to describe the system and can be obtained from an underlying relativistic mean field theory (see [36] for the multi-component single fluid, [37] for the two-component multifluid and appendix A for an extension to the three-component case) its thermodynamic interpretation is not obvious. In this subsection we assign a precise thermodynamic meaning to Λ . In doing this, we show that all the thermodynamic potentials usually employed in single-fluid thermodynamics can be recovered, and extended, to the multifluid case.

⁸ Clearly, $\mathcal{Q} \subset \mathcal{Z}$. For a given temperature of the bath, the equilibration process projects a point in \mathcal{Z} on \mathcal{Q} .

Consider the set \mathcal{Q} of all the possible equilibrium macrostates in which we can prepare the multifluid by setting the temperature of the bath⁸. The natural full set of variables which we can use to label a point in \mathcal{Q} is

$$(\Theta_H, N_A, N_i, W_j^i)_{i=1, \dots, k}^{A=1, \dots, l}. \quad (47)$$

This makes \mathcal{Q} a manifold of dimension $1 + l + 4k$. Assuming proper convexity properties for Λ and given that the volume V is a constant, the map

$$(\Theta_H, N_A, N_i, W_j^i)_{i=1, \dots, k}^{A=1, \dots, l} \mapsto (n_A^0, n_i^0, n_i^j)_{i=1, \dots, k}^{A=0, \dots, l} \quad (48)$$

is a bijection, meaning that both sets of variables can be used to label a point in \mathcal{Q} . In particular, if we now consider Λ as a function from \mathcal{Q} to \mathbb{R} , its differential has the form

$$d\Lambda = \sum_{A=0}^l \mu_0^A dn_A + \sum_{i=1}^k \mu_0^i dn_i^0 + \sum_{i=1}^k \mu_j^i dn_i^j, \quad (49)$$

where, since the normal components are at rest in the frame of the bath, $n_A^0 = n_A$ (no confusion should arise between indexes i and j because the sum over the former is always explicit and for the latter we use the Einstein convention). The internal energy density \mathcal{U} of the multifluid in thermal equilibrium with a bath is defined to be the energy density T^{00} (i.e. measured in the reference frame of the bath) seen as a function on \mathcal{Q} . Combining (17), (18), (35) and (36) it is immediate to show that $\mathcal{U} = T^{00}$ can be written as the Legendre transform of Λ with respect to spatial part of the superfluid currents,

$$\mathcal{U} = -\Lambda + \sum_{i=1}^k \mu_j^i n_i^j. \quad (50)$$

This implies that the natural set variables upon which \mathcal{U} depends is

$$(n_A, n_i^0, \mu_j^i)_{i=1, \dots, k}^{A=0, \dots, l} \quad (51)$$

and its differential, employing the definitions (38) and (44), reads

$$d\mathcal{U} = \Theta ds + \sum_{A=1}^l \mu_A^{(T)} dn_A + \sum_{i=1}^k \mu_i^{(T)} dn_i^0 + \sum_{i=1}^k n_i^j d\mu_j^i. \quad (52)$$

This formula alone contains the essence of all the results we collected up to now: since we can associate to any superfluid current a conjugate variable that is conserved during the equilibration process (the momentum), at a thermodynamic level we need to include a dependence on it in the equation of state. Moreover, the conservation of the superfluid momentum is the reason why, to get the temperature Θ or the thermodynamic chemical potentials $\mu_x^{(T)}$, the superfluid momenta must be fixed when performing the partial derivatives of \mathcal{U} .

Now that we identified the internal energy and its natural set of variables, we can make further Legendre transformations to define other thermodynamic potentials. If we transform with respect to the entropy density we get the Helmholtz free energy density

$$\mathcal{F} = \mathcal{U} - \Theta s, \quad (53)$$

whose differential (on the manifold \mathcal{Q}) is

$$d\mathcal{F} = -sd\Theta + \sum_{A=1}^l \mu_A^{(T)} dn_A + \sum_{i=1}^k \mu_i^{(T)} dn_i^0 + \sum_{i=1}^k n_i^j d\mu_j^i. \quad (54)$$

Notice that the function \mathcal{F} defined in (25) reduces to the Helmholtz free energy when the equilibrium is reached, which justifies the use of the same name.

We can also introduce the grand potential density

$$\mathcal{K} = \mathcal{U} - \sum_x n_x^0 \mu^{x0}, \quad (55)$$

which, considering equations (50) and (18), is found to be equal to $-\Psi$. Its differential is

$$d\mathcal{K} = -sd\Theta - \sum_{A=1}^l n_A d\mu_A^{(T)} - \sum_{i=1}^k n_i^0 d\mu_i^{(T)} + \sum_{i=1}^k n_i^j d\mu_j^i. \quad (56)$$

If in the above equation we substitute \mathcal{K} with $-\Psi$ we get the generalization of the Gibbs–Duhem relation for multifluids.

The presence of superfluid components extends the set of possible thermodynamic potentials. For example, the Legendre transform of the internal energy with respect to all the superfluid momenta reads

$$\mathcal{E} = \mathcal{U} - \sum_{i=1}^k n_i^j \mu_j^i, \quad (57)$$

whose differential is

$$d\mathcal{E} = \Theta ds + \sum_{A=1}^l \mu_A^{(T)} dn_A + \sum_{i=1}^k \mu_i^{(T)} dn_i^0 - \sum_{i=1}^k \mu_j^i dn_i^j. \quad (58)$$

Comparing equations (50) and (57) we obtain

$$\mathcal{E} = -\Lambda. \quad (59)$$

Despite its simplicity, this result plays a central role in our thermodynamic analysis, as it unveils the thermodynamic meaning of the master function: it is obtainable by means of a Legendre transformation of the internal energy density \mathcal{U} with respect to the superfluid momenta. From the practical point of view, this procedure is important because it can be used to reconstruct the master function directly from an equation of state for the internal energy (or from any other thermodynamic potential supplied by a microphysical calculation).

For the sake of completeness we present a further thermodynamic potential,

$$\mathcal{J} = \mathcal{U} - \sum_{i=1}^k n_i^0 \mu^{i0}, \quad (60)$$

whose thermodynamic differential is

$$d\mathcal{J} = \Theta ds + \sum_{A=1}^l \mu_A^{(T)} dn_A - \sum_{i=1}^k n_i^0 d\mu_i^{(T)} + \sum_{i=1}^k n_i^j d\mu_j^i. \quad (61)$$

The potential \mathcal{J} depends on an interesting set of state variables. In microphysical calculations, to define the phases ϕ^i of the order parameters, one is usually forced to break the conservation of the number of particles of the superfluid species [32]. This suggests to work with an

ensemble in which the chemical potential $\mu_i^{(T)}$ (which by Josephson relation (30) is proportional to $\partial_i \phi^j$) is assigned and n_i^0 is computed as a quantum-statistical average. For this reason \mathcal{J} is a natural alternative to \mathcal{U} in the construction of an equation of state.

3.5. The Euler relation

Like in the thermodynamic study of a simple fluid, it is possible to derive the Euler (or fundamental) relation also in the multifluid case (see e.g. the discussion in [4]). Since the Euler relation stems from the additive property of the system, its derivation is typically performed by employing the thermodynamic potential which is a natural function of all the extensive quantities. In our case this thermodynamic potential is \mathcal{E} and equation (18) implies that the fundamental relation is

$$\mathcal{E} = -\Psi + \sum_x n_x^0 \mu^{x0} - \sum_{i=1}^k n_i^j \mu_j^i. \quad (62)$$

It is interesting to recast the fundamental relation in terms of the internal energy:

$$\mathcal{U} = -\Psi + \Theta s + \sum_{A=1}^l n_A \mu_A^{(T)} + \sum_{i=1}^k n_i^0 \mu_i^{(T)}. \quad (63)$$

Since it does not explicitly contain any term associated with the presence of superfluids in the right-hand side, this form of the Euler relation is formally identical to the one for a mixture of normal fluids. This is due to the fact that the set of natural variables of \mathcal{U} , presented in (51), contains the spatial components of the momenta per particle, which are intrinsically intensive quantities and, therefore, cannot produce the associated terms in the Euler relation.

It is possible to write the fundamental relation (62) also in a manifestly covariant way that is, however, less physically transparent. For example, in [38], the name ‘chemical potentials’ is assigned to the quantities

$$\mu_x^{(C)} = -\mu_\nu^x u_x^\nu, \quad (64)$$

where u_x^ν is the four-velocity associated to the species x . We use the label C , here, which stands for *comoving*, to distinguish them from those defined in (44), in which we use a T , for *thermodynamic*. According to this definition, the Euler relation becomes (see also [39])

$$\mathcal{E} = -\Psi + \Theta s + \sum_{x \neq s} n_x \mu_x^{(C)}. \quad (65)$$

Notice that (65) and (63) are both very similar to the usual Euler relation for a simple fluid. However, one must not be tempted to identify \mathcal{E} with the thermodynamic energy density as this would lead to the confusion between the thermodynamic and the comoving chemical potentials, which can assume very different values. An interpretation problem related to the different chemical potentials used in the two-fluid models of [23] and [24] is discussed and solved in appendix C.

3.6. Entrainment and thermodynamics

One of the properties of a multifluid is the possibility to have non-collinearity between the currents and their conjugate momenta, which gives rise to the so-called entrainment effect. In

the formulation presented in section 2 the momenta naturally arise as linear combinations of the currents, see (16):

$$\mu_\nu^x = \sum_y \mathcal{B}^{xy} n_{y\nu}, \quad (66)$$

where we have introduced the symmetric matrix \mathcal{B}^{xy} such that

$$\mathcal{B}^{xx} := \mathcal{B}^x \quad \mathcal{B}^{xy} := \mathcal{A}^{xy} \quad (67)$$

for $x \neq y$. On the other hand, if we define the symmetric matrix \mathcal{B}_{xy} to be the inverse of \mathcal{B}^{xy} ,

$$\sum_y \mathcal{B}^{xy} \mathcal{B}_{yz} = \delta_z^x, \quad (68)$$

we can write the currents as linear combinations of the momenta:

$$n_x^\nu = \sum_y \mathcal{B}_{xy} \mu^{y\nu}. \quad (69)$$

This relationship between momenta and currents is the most natural when we start from the momentum-based ‘potential variational principle’ instead of the ‘current-based’ one of section 2. As pointed out by [30], see also [40] and [41], in the potential variational principle the master function of the theory is $\Psi = -\mathcal{K}$ written as a function of the scalar products $\mu_{xy}^2 = -\mu_\nu^x \mu^{y\nu}$, in place of $\Lambda = -\mathcal{E}$ written as a function of n_{xy}^2 .

In a thermodynamic perspective, we are interested in writing the normal momenta and the superfluid currents as linear combinations of the normal currents and of the superfluid momenta. The reason for this is that the normal currents identify the reference frame in which the thermodynamic quantities must be computed, while the superfluid momenta enter directly into the equation of state as intensive variables. This representation of the entrainment arises naturally from a hybrid variational principle which uses $\mathcal{X} = -\mathcal{J}$ as the master function, see e.g. [30, 42, 43]. The entrainment matrix for this hybrid description can be written with a little algebra in terms of \mathcal{B}_{xy} , as pointed out in [40]. It is now important to briefly review and extend this set of ideas.

To introduce the hybrid representation, let us separate the normal components from the superfluid ones in equation (69), namely

$$n_A^\nu = \sum_{B=0}^l \mathcal{V}_{AB} \mu^{B\nu} + \sum_{h=1}^k \mathcal{M}_{Ah} \mu^{h\nu} \quad n_i^\nu = \sum_{B=0}^l \mathcal{M}_{Bi} \mu^{B\nu} + \sum_{h=1}^k \mathcal{S}_{ih} \mu^{h\nu}, \quad (70)$$

where $A = 0$ stands for s . The three matrices appearing in the right hand side are respectively the $(1+l) \times (1+l)$ symmetric matrix

$$\mathcal{V}_{AB} = \mathcal{B}_{AB},$$

the $k \times k$ symmetric matrix

$$\mathcal{S}_{ih} = \mathcal{B}_{ih}$$

and the $(1+l) \times k$ matrix

$$\mathcal{M}_{Ai} = \mathcal{B}_{Ai}. \quad (73)$$

With a little algebra it is possible to show that

$$\mu_\nu^A = \sum_{B=0}^l (\mathcal{V}^{-1})^{AB} n_{B\nu} - \sum_{h=1}^k \mathcal{D}_h^A \mu_\nu^h, \quad n_i^\nu = \sum_{B=0}^l \mathcal{D}_i^B n_B^\nu + \sum_{h=1}^k Y_{ih} \mu^{h\nu}, \quad (74)$$

where we have introduced the $(1+l) \times k$ matrix

$$\mathcal{D}_i^A = \sum_{B=0}^l (\mathcal{V}^{-1})^{AB} \mathcal{M}_{Bi} \quad (75)$$

and the $k \times k$ symmetric matrix

$$Y_{ih} = \mathcal{S}_{ih} - \sum_{B,C=0}^l (\mathcal{V}^{-1})^{BC} \mathcal{M}_{Bi} \mathcal{M}_{Ch}. \quad (76)$$

With calculations analogous to those made to obtain the system (74), the superfluid block of the entrainment matrix given in (66) is

$$\mathcal{B}^{ih} = (\mathcal{Y}^{-1})^{ih}. \quad (77)$$

The matrix Y_{ih} naturally encodes the notion of entrainment in a thermodynamic context. To give a precise meaning to this sentence, let us consider again equation (52). Combining (74), (35) and (36) we get

$$n_i^j = \sum_{h=1}^k Y_{ih} \mu^{hj}, \quad (78)$$

which, using the symmetry of Y_{ih} , implies

$$d\mathcal{U} = \Theta ds + \sum_{A=1}^l \mu_A^{(T)} dn_A + \sum_{i=1}^k \mu_i^{(T)} dn_i^0 + \sum_{i,h=1}^k \frac{Y_{ih}}{2} d(\mu^{hj} \mu_j^i). \quad (79)$$

This tells us that, in a thermodynamic perspective, the interesting part of the entrainment matrix is the one that defines the relationship between the superfluid momenta and the superfluid currents in the reference frame of the heat bath. In fact, in the following section we show that Y_{ih} is the only physically important part of the entrainment matrix in a non-dissipative theory, while \mathcal{V}_{AB} and \mathcal{D}_i^A have a microscopic interpretation only if one takes into account dissipative effects, so their knowledge is not needed and cannot be uniquely extracted within the framework of equilibrium thermodynamics.

4. Local thermodynamics of perfect multifluids

In this section we seek a formulation of multifluid hydrodynamics in which the thermodynamic interpretation of the variables is manifest. This is certainly possible for a perfect multifluid since, in the absence of dissipation, the second principle ensures that the system is in local thermodynamic equilibrium. Hence, the formalism developed in section 3 should be locally valid in a frame comoving with the entropy.

Before getting into the details, let us comment on how the equations of motion of the phenomenological model of section 2.2 are modified in view of the results of the previous section. Clearly the normal currents must be implemented as s -locked currents, while the superfluid

ones as free. Moreover, superfluidity imposes the irrotationality of the momenta. Hence, the full system of hydrodynamic equations for a perfect multifluid is

$$\nabla_\nu n_x^\nu = 0 \quad \partial_{[\rho} \mu_{\nu]}^i = 0 \quad \nabla_\nu T^\nu_\rho = 0. \quad (80)$$

This system is compatible with equation (19), the only difference is that the irrotationality requirement adds a further constraint to the initial condition.

4.1. The local frame of thermodynamics

It is instructive to rework the Lagrangian approach of section 2 with the aid of a tetrad comoving with s^μ , i.e. a collection of four vector fields $e_a = e_a^\nu \partial_\nu$, $a = 0, 1, 2, 3$ that are orthonormal⁹ and such that $e_0 = u$. This tetrad represents the local Minkowskian frame of an observer \mathcal{O} moving with four-velocity u . The components of a generic tensor on this basis can be interpreted as the physical quantities seen by \mathcal{O} (see e.g. [44]).

The physical motivation for considering this particular tetrad is that the normal currents observed by \mathcal{O} are $n_A^j = 0$, for $A = 0, \dots, l$. Therefore, a perfect multifluid is *locally* in a state belonging to \mathcal{Q} , the manifold of equilibrium macrostates defined in section 3.4. This implies that the energy density measured by \mathcal{O} coincides with the thermodynamic variable \mathcal{U} , namely

$$u_\nu u_\rho T^{\nu\rho} = \mathcal{U}. \quad (81)$$

Now, equations (38) and (44) define the local temperature and chemical potentials of the multifluid; in the tetrad formalism they simply become

$$\Theta = \Theta^0 \quad \mu_x^{(T)} = \mu^{x0}. \quad (82)$$

For later convenience we also introduce the two orthogonal projectors

$$\parallel^\nu_\rho := -u^\nu u_\rho = e_0^\nu e_\rho^0 \quad \perp^\nu_\rho := \delta^\nu_\rho + u^\nu u_\rho = e_j^\nu e_\rho^j. \quad (83)$$

Since these two projectors are complementary, i.e.

$$\delta^\nu_\rho = \parallel^\nu_\rho + \perp^\nu_\rho, \quad (84)$$

they can be used to separate the time and the space parts of a vector in the tetrad frame: the tensor \parallel^ν_ρ projects any vector along u , while \perp^ν_ρ projects it tangentially to the local present of the observer \mathcal{O} .

4.2. The emergence of the Landau–Andreev–Bashkin formalism

We now use the tetrad to endow the non-homogeneous phenomenological model with a language that is more directly in contact with the microphysics of the system. In pursuing this aim, we find that such a language is the one proposed by Gusakov (see [10]), which is the multi-component generalization of the single-component superfluid model proposed by Son in [21]. While the work of Son provides a natural relativistic extension of the non-relativistic two-fluid model pioneered by Tisza and Landau (see e.g [45]), the formalism developed by

⁹The fields e_a are assumed to be linearly independent everywhere, so that we can define the matrix e_ν^a to be the inverse of e_a^ν , which means $e_\nu^a e_b^\nu = \delta_b^a$ and $e_a^\nu e_\nu^a = \delta^\nu_\rho$. Moreover, they satisfy the orthonormality condition $g(e_a, e_b) = \eta_{ab}$. In the following, for the tetrad indices we will use the convention that the generic latin indices a and b run over all the possible indices (from 0 to 3), while $j = 1, 2, 3$ are restricted to the spatial ones, in accordance with the convention introduced in the previous section. Einstein summation convention will be applied to both a and j indices.

Gusakov can be considered as the natural formulation in general relativity of the multifluid hydrodynamics pioneered by [29]. For this reason, in the following, we will speak of relativistic Landau–Andreev–Bashkin (LAB) formalism, as opposed to the phenomenological Carter formalism of section 2. Here we show how the LAB formalism naturally emerges from the Carter’s approach in view of the thermodynamic considerations of section 3.

Let us define the *winding vector*¹⁰ to be the spatial part of the superfluid momentum relative to a superfluid species i , namely

$$w_{i\nu} := \perp_{\nu}^{\rho} \mu_{\rho}^i = \mu_j^i e_{\nu}^j. \quad (85)$$

With the aid of (84), the superfluid momenta μ_{ν}^i are split as

$$\mu_{\nu}^i = \mu_i^{(T)} u_{\nu} + w_{i\nu}. \quad (86)$$

On the other hand, the same operation performed on the superfluid currents n_i^{ν} gives

$$n_i^{\nu} = n_i^{(T)} u^{\nu} + \sum_h Y_{ih} w_h^{\nu}, \quad (87)$$

where we have employed equation (74) and the definition

$$n_i^{(T)} := -u_{\nu} n_i^{\nu}, \quad (88)$$

which is the density of the species i measured by \mathcal{O} .

Finally, we split the energy–momentum tensor as

$$T^{\nu\rho} = T^{ab} e_a^{\nu} e_b^{\rho} = T^{00} e_0^{\nu} e_0^{\rho} + T^{j0} (e_0^{\nu} e_j^{\rho} + e_j^{\nu} e_0^{\rho}) + T^{jj'} e_j^{\nu} e_{j'}^{\rho}. \quad (89)$$

Thanks to equation (81) and the fact that $n_A^j = 0$ in the tetrad frame, we can identify the various components of T^{ab} as

$$T^{00} = \mathcal{U} \quad T^{j0} = \sum_{i=1}^k n_i^j \mu^{i0} \quad T^{jj'} = \Psi \eta^{jj'} + \sum_{i=1}^k n_i^j \mu^{ij'}, \quad (90)$$

which leads to

$$T^{\nu\rho} = (\mathcal{U} + \Psi) u^{\nu} u^{\rho} + \Psi g^{\nu\rho} + \sum_{i,h=1}^k Y_{ih} (\mu_i^{(T)} u^{\nu} w_h^{\rho} + \mu_h^{(T)} u^{\rho} w_i^{\nu} + w_i^{\nu} w_h^{\rho}). \quad (91)$$

Interestingly, this reworking of the energy momentum tensor makes evident that a mixture of normal components (i.e. when $k = 0$) is equivalent to a perfect fluid. We now can combine (79) with (85) and (88) to get an expression for the differential of \mathcal{U} that involves only variables that are manifestly covariant,

$$d\mathcal{U} = \Theta ds + \sum_{A=1}^l \mu_A^{(T)} dn_A + \sum_{i=1}^k \mu_i^{(T)} dn_i^{(T)} + \sum_{i,h=1}^k \frac{Y_{ih}}{2} d(w_i^{\nu} w_{h\nu}). \quad (92)$$

One can verify that (1), (2), (87), (86), (91), (92) and (80) define a complete (and also redundant) self-consistent system of hydrodynamic equations. Hence, for a generic perfect multifluid the LAB formalism is completely equivalent to the Lagrangian approach of Carter.

¹⁰ Following Landau’s original terminology, in [10] and [22] the quantity w_i^{ν} is referred to as a ‘four-velocity’ for the motion of a superfluid component. Since w_i^{ν} is not normalized to -1 , we prefer to call it ‘winding vector’ in view of equation (33).

In particular, the equivalence between the Son's relativistic version of the Landau two-fluid model and the Carter's formalism has already been proven in [4]; here we generalised the discussion by considering the most general theory for a perfect multifluid and we provided a dictionary to easily translate the two descriptions one into the other.

4.3. The problem of the redundancy of the master function

We cast the perfect multifluid hydrodynamics in a way that only the knowledge of \mathcal{U} (or any other thermodynamic potential) is required. However, given the expression for \mathcal{U} , one would like to also have a technique to write the corresponding master function.

Since $\Lambda = -\mathcal{E}$, the first obvious step is to obtain \mathcal{E} from \mathcal{U} via the Legendre transform (57). However, this is not sufficient to complete the task: Λ is defined as a function of the scalars n_{xy}^2 , while by making the Legendre transform of \mathcal{U} we arrive at $\mathcal{E} = \mathcal{E}(n_x^0, n_i^j)$. Rewriting the dependencies correctly may seem straightforward but there is a subtlety which arises from the fact that some of the components are s -locked. Before going into the details it is convenient to give an idea of where the complication comes from.

Consider a simple perfect fluid at finite temperature, i.e. $l = 1$ and $k = 0$. The two currents of the theory are

$$s^\nu = su^\nu \quad n^\nu = nu^\nu, \quad (93)$$

where n^ν is the (s -locked) current of particles. The energy-momentum tensor can be written according to the the LAB formalism by setting $k = 0$ into (91),

$$T^{\nu\rho} = (\mathcal{U} + \Psi)u^\nu u^\rho + \Psi g^{\nu\rho}. \quad (94)$$

According to the LAB formulation, the only microscopic input needed to describe a perfect fluid is an equation of state of the form

$$\mathcal{U} = \mathcal{U}(s, n), \quad (95)$$

whose differential is

$$d\mathcal{U} = \Theta ds + \mu^{(T)} dn. \quad (96)$$

Now, let us consider the master function

$$\Lambda = \Lambda(n_{ss}^2, n_{sn}^2, n_{nn}^2). \quad (97)$$

Combining (50), (35) and (36), it is immediate to find that

$$\mathcal{U}(s, n) = -\Lambda(s^2, sn, n^2). \quad (98)$$

Therefore, to construct a function \mathcal{U} that fully describes the fluid, only the behaviour of Λ on the surface

$$n_{sn}^2 = \sqrt{n_{ss}^2 n_{nn}^2}, \quad (99)$$

is needed. On the other hand, the values of Λ elsewhere constitute a part of the information which is completely lost when the Son and Gusakov formulation of hydrodynamics is used. In other words, two different master functions Λ and Π which are equal on this surface would

produce the same LAB description and, therefore, the same physical theory. This redundancy in the arguments of the master function arises because of the collinearity requirement for s^ν and n^ν : the two functions Λ and Π are equal on all the possible equilibrium macrostates.

The fact that Λ and Π describe the same physical system implies that there is not a unique prescription to define \mathcal{V}_{AB} : there are many equivalent ways to construct the entrainment matrix. This arbitrariness is a peculiarity of the non-dissipative theory. In fact, the behaviour of Λ on the macrostates in which s^ν and n^ν are not locked becomes relevant for dissipative models with heat transport. In fact, for dissipative systems the entrainment with the entropy component becomes a necessary and physically meaningful element of the description (see e.g. [14]). On the contrary, in the non-dissipative limit this arbitrariness may be advantageous as it allows to set the entrainment anomaly \mathcal{A}^{sn} to zero imposing

$$\Lambda = -\mathcal{U}(\sqrt{n_{ss}^2}, \sqrt{n_{nn}^2}). \quad (100)$$

In this simple case with $k = 0$ it does not seem to make any difference the fact that the entrainment matrix is not uniquely defined: since there are no superfluid components, the currents are all parallel with respect to each other and any master function we choose will produce in the end the same momenta

$$\Theta_\nu = \Theta u_\nu \quad \mu_\nu = \mu^{(T)} u_\nu. \quad (101)$$

However, in the case of a generic perfect multifluid, not only \mathcal{V}_{AB} but also \mathcal{D}^A_i is not uniquely defined; a change of the master function can modify the definition of the normal momenta. The following three subsections are devoted to the analysis of this non-uniqueness of the normal momenta. In section 4.4 we study the redundancy problem of the master function starting from the action principle, in section 4.5 we analyse its physical interpretation in a thermodynamic perspective. Finally, in section 4.6 we provide a practical example in which we show how to fix the master function to remove the entrainment associated to the entropy in neutron star matter.

4.4. Mathematical explanation of the redundancy

Let us consider, for a generic multifluid, two different functions Λ and Π from \mathbb{R}^z to \mathbb{R} . The z real arguments of these functions are the numbers n_{xy}^2 , considered as independent variables. If the normal species are forced to comove with the entropy, i.e. when

$$n_A^\nu = n_A \frac{s^\nu}{s}, \quad (102)$$

the arguments of Λ and Π are no more free, but satisfy the conditions

$$n_{Ax}^2 = \frac{n_A}{s} n_{sx}^2. \quad (103)$$

Assume that Λ and Π are equal when the above constraints are satisfied but that are different on an out of local equilibrium macrostate of the multifluid. Remembering that the normal currents comove with the entropy both on and off shell, according to equation (12) we have

$$\begin{aligned} \mathcal{L}[g, X_s^\alpha, X_i^\alpha] &= \Lambda(-g_{\rho\nu} n_x^\rho[g, X_s^\alpha, X_i^\alpha] n_y^\nu[g, X_s^\alpha, X_i^\alpha]) \\ &\equiv \Pi(-g_{\rho\nu} n_x^\rho[g, X_s^\alpha, X_i^\alpha] n_y^\nu[g, X_s^\alpha, X_i^\alpha]). \end{aligned} \quad (104)$$

This means that, since the energy–momentum tensor and all the hydrodynamic equations are calculated only imposing the stationarity of the action, the choice between Λ and Π does not affect the physics of the perfect multifluid, in the sense that they give rise to the same physical theory: both can be used to construct the same equations of motion and energy–momentum tensor, which can then be used to uniquely derive the LAB description of the system by using the method of section 4.2.

However, the choice between Λ and Π can modify concretely the normal momenta. To see this, consider a perfect multifluid on shell, choose a point of the spacetime and compute the component μ_j^A in the basis introduced in section 4.1, for $A \neq s$, directly from the master function. By definition $n_A^j = 0$, so that

$$\mu_j^A = \left. \frac{\partial \Lambda}{\partial n_A^j} \right|_{n_A^j=0} = \lim_{h \rightarrow 0} \frac{\Lambda(n_A^j = h) - \Lambda(n_A^j = 0)}{h}, \quad (105)$$

where all the remaining n_x^a are kept fixed. Since the derivative is computed for $s^j = 0$, the quantity $\Lambda(n_A^j = h)$ is the master function evaluated on a macrostate in which the currents n_A^ν and s^ν are not parallel with respect to each other: performing the above derivative implies that we are exploring a domain in which Λ may differ from Π .

The freedom to choose the master function on the macrostates in which one (or more) s -locked current does not comove with the entropy can be seen as a gauge freedom of the non-dissipative theory. Only performing a gauge fixing (i.e. assigning the master function in a neighbourhood of the equilibrium) it is possible to define uniquely the normal momenta. Notice that only the spatial part $\perp_\nu \mu_\rho^A$ of the normal momenta is affected by a gauge-fixing of the master function. In fact, the previous argument is no more valid for μ_0^A , because the variation of the argument is performed in the direction of the entropy:

$$\mu_0^A = \frac{\partial \Lambda}{\partial n_A^0} = \lim_{h \rightarrow 0} \frac{\Lambda(n_A^j = 0, n_A^0 + h) - \Lambda(n_A^j = 0, n_A^0)}{h}. \quad (106)$$

Also the superfluid momenta are uniquely determined, because in the corresponding derivation we have to move the superfluid current keeping the normal ones collinear. This justifies the statement, made in section 3.6, that in a non-dissipative theory Y_{ih} is the only physically meaningful part of the entrainment matrix. Stated in other words, according to equation (77) only the block \mathcal{B}^{ih} of the full entrainment matrix \mathcal{B}^{xy} is gauge-independent.

Since the choice between Λ and Π does not affect the energy–momentum tensor, the transformations $\mu_j^A \rightarrow \tilde{\mu}_j^A$ produced in the passage from Λ to Π , are not completely free, but must satisfy some constrains. In fact, the currents produced by considering the two different gauge-fixing must satisfy

$$s^0 \Theta_j + \sum_{A=1}^l n_A^0 \mu_j^A + = s^0 \tilde{\Theta}_j + \sum_{A=1}^l n_A^0 \tilde{\mu}_j^A. \quad (107)$$

This can be proven by considering the general expression of $T^{\nu\rho}$ in (17), together with the fact that the energy–momentum tensor (in particular T_j^0) and the superfluid momenta are gauge-invariant quantities. We can use the above constraint to make explicit what happens when a gauge transformation is performed:

$$\tilde{\mu}_\nu^i = \mu_\nu^i \quad \tilde{\mu}_\nu^A = \mu_\nu^A + \pi_\nu^A \quad \tilde{\Theta}_\nu = \Theta_\nu + \zeta_\nu, \quad (108)$$

where

$$s^\nu \pi_\nu^A = 0 \quad \zeta_\nu = -\frac{1}{s} \sum_{A=1}^l n_A \pi_\nu^A. \quad (109)$$

Therefore, the gauge transformation arising from a different choice of the master function amounts to a redistribution of the spatial part of the momentum between the normal currents.

Finally, let us remark that the gauge-freedom discussed here should not be confused with the notion of *chemical gauge* introduced in [26], as discussed in appendix B.

4.5. Gauge-invariant quantities

The fact that two different master functions (say the Λ and Π of the previous subsection) may produce the same LAB description of a given multifluid system can be used to understand in a physical perspective why some quantities are invariant under the gauge transformation (108). Let us examine them one by one.

According to (108), the superfluid momenta are not affected by a gauge fixing of the master function. It is therefore possible to interpret them as gradients of the phases ϕ^i (i.e. to assign them a physical interpretation that is beyond the Carter's phenomenological approach).

Now, consider equations (91) and (92), which define the natural variables of the LAB description. Equation (108) tells us that the differentiated variables (i.e. n_A , $n_i^{(T)}$ and w_i^ν) in the right-hand side of (92) are gauge-invariant: since \mathcal{U} is gauge invariant, so are also all its derivatives, namely $\Theta = -\Theta_\nu u^\nu$, $\mu_x^{(T)} = -\mu_\nu^x u^\nu$ and Y_{ih} . In particular, the fact that Θ and $\mu_x^{(T)}$ are gauge-invariant further justifies their interpretation as the absolute temperature and the chemical potentials of the system.

Also the generalized pressure $\Psi = -\mathcal{K}$ is a gauge invariant, as it can be obtained from \mathcal{U} via Legendre transform. In brief, all the quantities that naturally appear in the LAB description must be gauge-invariant (i.e. are unaffected by a gauge fixing of the master function).

Finally, the normal currents do not explicitly appear in (92). Even if equation (17) seems to suggest that the knowledge of μ_j^A is required to construct the energy–momentum tensor, there is no need to specify them, due to the fact that the symmetry of the energy–momentum tensor $T^{0j} = T^{j0}$ yields

$$\sum_{A=0}^l n_A^0 \mu_j^A + \sum_{i=1}^k n_i^0 \mu_j^i = \sum_{i=1}^k \mu^{i0} n_{ij}. \quad (110)$$

This proves that the only part of the normal momenta whose knowledge is explicitly required is the thermodynamic chemical potential.

To give a physical interpretation of this kind of gauge freedom we may start interpreting μ_j^A as the average momentum per particle of the species A . Now, without a microphysical insight it is difficult, in equilibrium thermodynamics, to separate the contribution which comes from the elementary excitations and should strictly contribute to Θ_j from the genuine momentum of the species A . This ambiguity describes the absence of an entirely *macroscopic* criterion to distribute the momentum between the normal components in the non-dissipative limit of the theory: the only prescription to uniquely assign the entrainment matrix is to consider a dissipative theory in which the normal currents are not forced to comove with the entropy. In this case, the heat contribution to the energy–momentum tensor contributes to fix the entrainment matrix, see e.g. [14].

On the other hand, if the purpose is to construct a theory for a given perfect multifluid, it is possible to take advantage of this fundamental gauge freedom to choose a master function that is particularly convenient for the specific system under study.

4.6. A practical example of gauge fixing

The minimal model of superfluid neutron stars consists of two distinct interpenetrating dynamical components, one normal ($l = 1$) and one superfluid ($k = 1$), see e.g. [17]. Following [18], the superfluid current n'_n is comprised of free neutrons, while the normal current n'_p can be defined as the remaining part of the conserved total baryon current. In this subsection we exploit the gauge freedom to show that there is no loss of generality in tuning such a minimal model in a way that the entropy does not carry any entrainment, namely

$$\perp^\rho_\nu \Theta_\rho = 0. \quad (111)$$

Consider the generic master function

$$\Lambda = \Lambda(s^2, n_{pp}^2, n_{nn}^2, n_{sp}^2, n_{sn}^2, n_{pn}^2), \quad (112)$$

so that the most general form of the momenta is

$$\begin{aligned} \mu_\nu^n &= \mathcal{B}^n n_{n\nu} + \mathcal{A}^{sn} s_\nu + \mathcal{A}^{pn} n_{p\nu} \\ \mu_\nu^p &= \mathcal{B}^p n_{p\nu} + \mathcal{A}^{sp} s_\nu + \mathcal{A}^{pn} n_{n\nu} \\ \Theta_\nu &= \mathcal{C} s_\nu + \mathcal{A}^{sp} n_{p\nu} + \mathcal{A}^{sn} n_{n\nu}. \end{aligned} \quad (113)$$

In the absence of dissipation we must impose $s_\nu = s n_{p\nu}/n_p$ and the above definitions become

$$\begin{aligned} \mu_\nu^n &= \mathcal{B}^n n_{n\nu} + \left(\mathcal{A}^{sn} \frac{s}{n_p} + \mathcal{A}^{pn} \right) n_{p\nu} \\ \mu_\nu^p &= \left(\mathcal{B}^p + \mathcal{A}^{sp} \frac{s}{n_p} \right) n_{p\nu} + \mathcal{A}^{pn} n_{n\nu} \\ \Theta_\nu &= \left(\mathcal{C} \frac{s}{n_p} + \mathcal{A}^{sp} \right) n_{p\nu} + \mathcal{A}^{sn} n_{n\nu}. \end{aligned} \quad (114)$$

Therefore, at this level every conjugate momentum is expected to depend on all the currents of the theory. However, a convenient gauge fixing can be performed as follows: the s-locking property of the normal component implies

$$n_{sp}^2 = \sqrt{s^2 n_p^2} \quad n_{sn}^2 = \frac{n_{pn}^2 \sqrt{s^2}}{\sqrt{n_p^2}}, \quad (115)$$

so that it is possible to define an alternative master function Π as

$$\Pi(s^2, n_{pp}^2, n_{nn}^2, n_{pn}^2) := \Lambda\left(s^2, n_{pp}^2, n_{nn}^2, \sqrt{s^2 n_{pp}^2}, \frac{n_{pn}^2 \sqrt{s^2}}{\sqrt{n_{pp}^2}}, n_{pn}^2\right). \quad (116)$$

Under the hypothesis that the model is defined by the specification of Λ , the use of Π is completely admissible because both master functions produce same Lagrangian density \mathcal{L} . Now, the entrainment matrix is not gauge independent; in particular, the diagonal coefficients $\tilde{\mathcal{B}}^x$ obtained as derivatives of Π , are linked to the ones obtained from Λ via

$$\begin{aligned}\tilde{\mathcal{B}}^n &= -2 \frac{\partial \Pi}{\partial n_m^2} = \mathcal{B}^n, \\ \tilde{\mathcal{B}}^p &= -2 \frac{\partial \Pi}{\partial n_{pp}^2} = \mathcal{B}^p + \mathcal{A}^{sp} \frac{s}{n_p} - \mathcal{A}^{sn} \frac{s n_{pn}^2}{n_p^3}, \\ \tilde{\mathcal{C}} &= -2 \frac{\partial \Pi}{\partial s^2} = \mathcal{C} + \mathcal{A}^{sp} \frac{n_p}{s} + \mathcal{A}^{sn} \frac{n_{pn}^2}{s n_p}.\end{aligned}\quad (117)$$

Regarding the off-diagonal coefficients, Π does not depend on n_{sp}^2 and n_{sn}^2 , implying that

$$\tilde{\mathcal{A}}^{sp} = \tilde{\mathcal{A}}^{sn} = 0. \quad (118)$$

The entrainment is now described only in terms of a single anomaly coefficient:

$$\tilde{\mathcal{A}}^{pn} = -\frac{\partial \Pi}{\partial n_{pn}^2} = \mathcal{A}^{pn} + \mathcal{A}^{sn} \frac{s}{n_p}. \quad (119)$$

The calculation of the new momenta is straightforward and it is possible to verify that

$$\tilde{\mu}_\nu^n = \mu_\nu^n \quad \tilde{\mu}_\nu^p = \mu_\nu^p + \frac{s}{n_p} \mathcal{A}^{sn} \left[n_{n\nu} - \frac{n_{pn}^2 n_{p\nu}}{n_p^2} \right] \quad \tilde{\Theta}_\nu = \Theta_\nu - \mathcal{A}^{sn} \left[n_{n\nu} - \frac{n_{pn}^2 n_{p\nu}}{n_p^2} \right]. \quad (120)$$

The above system of equations is presented in the form (108), giving the immediate identification

$$\pi_\nu = \frac{s}{n_p} \mathcal{A}^{sn} \left[n_{n\nu} - \frac{n_{pn}^2 n_{p\nu}}{n_p^2} \right] \quad \zeta_\nu = -\frac{n_p}{s} \pi_\nu = -\mathcal{A}^{sn} \left[n_{n\nu} - \frac{n_{pn}^2 n_{p\nu}}{n_p^2} \right]. \quad (121)$$

The result of the gauge transformation in (120) can be written in a more elegant way with the aid of the two complementary projectors defined in (83),

$$\tilde{\mu}_\nu^n = \mu_\nu^n \quad \tilde{\mu}_\nu^p = \mu_\nu^p + \frac{s}{n_p} (\perp \Theta)_\nu \quad \tilde{\Theta}_\nu = (\parallel \Theta)_\nu. \quad (122)$$

Thanks to this more geometrically transparent form, it is easy to notice that, since

$$\pi_\nu = \frac{s}{n_p} (\perp \Theta)_\nu, \quad (123)$$

also the first relation of (109) is automatically satisfied, as it should be. Moreover, from the last expression in (122), it is also evident that we have redefined the entrainment to satisfy (111). This completes the proof that, given an arbitrary Λ and provided that $n_p \neq 0$, it is always possible to make a gauge-fixing which removes the entrainment from the entropy.

Finally, it is possible to verify with direct calculations that Θ , $\mu_x^{(T)}$, Y_m , Ψ , $T^{\nu\rho}$ and the hydrodynamic equations are left unchanged by the above transformations. This proves that Λ and Π produce the same hydrodynamic model when the LAB formalism is used.

The importance of this example is represented by the fact that it shows how the gauge can be used to simplify the hydrodynamic model, removing the redundancies. A further practical application that exploits the gauge freedom of the system is presented at the end of section 5.1.

4.7. Constructing the master function from a given equation of state

We now present two simple examples of how to assign $\Lambda(n_{xy}^2)$ for a system described by a given equation of state. For simplicity, we assume to know \mathcal{E} (if the microscopic analysis of the system provides a different thermodynamic potential it is sufficient to perform a Legendre transformation).

Tisza–Landau two-fluid model ($l = 0, k = 1$)

[42] proposed a relativistic version of the Tisza–Landau model for Helium-II: we briefly discuss this model for a single component superfluid at finite temperature and provide a practical formula to set the master function.

The two currents of the theory are s^μ , describing the gas of excitations, and a free current n^μ . In general, the master function has the form

$$\Lambda = \Lambda(s^2, n^2, n_{sn}^2), \quad (124)$$

so that there is no gauge freedom here: n_{sn}^2 carries all the information about the relative speed between the two components. All we need to do is to rearrange the variables in order to make everything dependent only on three scalars that could be conveniently used in a thermodynamic potential. Let $n^\mu = n^0 e_0^\mu + n^1 e_1^\mu$ be the particle current expressed in the frame in which excitations have average zero velocity; the potential \mathcal{E} may be given as a function of three thermodynamic variables as

$$\mathcal{E} = \mathcal{E}(s^0, n^0, n^1). \quad (125)$$

Noticing that

$$s^0 = \sqrt{s^2} \quad n^0 = n_{sn}^2 / \sqrt{s^2}, \quad (126)$$

it is immediate to write n^1 in terms of the fundamental scalars s^2, n^2 and n_{sn}^2 as

$$n^1 = \sqrt{(n^0)^2 - n^2}. \quad (127)$$

We conclude that, given the thermodynamic potential in (125), a consistent hydrodynamic theory can be constructed starting from the master function

$$\Lambda(s^2, n^2, n_{sn}^2) = -\mathcal{E}\left(\sqrt{s^2}, \frac{n_{sn}^2}{\sqrt{s^2}}, \sqrt{\frac{(n_{sn}^2)^2}{s^2} - n^2}\right). \quad (128)$$

Alternatively, if the internal energy is provided, say $\mathcal{U} = \mathcal{U}(s, n^0, \mu_1)$, the first step to obtain the master function is to perform the Legendre transform $\mathcal{E} = \mathcal{U} - n^1 \mu_1$ and then to make the change of variables presented above.

Andreev and Bashkin model ($l = 0, k = 2$)

We now consider the mixture of two superfluid species at finite temperature. Such a system was early considered by [28] as a multicomponent generalization of the Tisza–Landau model,

refined by [29] for a superfluid ${}^3\text{He}$ - ${}^4\text{He}$ mixture and used to construct a Newtonian description of the matter in the outer core of a neutron star by [46]. We label the two species by p and n ; the corresponding currents, say n_p^ν and n_n^ν , are both free (again, there is no gauge freedom since $l = 0$). In this case, the general form of a master function is

$$\Lambda = \Lambda(s^2, n_{pp}^2, n_{nn}^2, n_{sp}^2, n_{sn}^2, n_{pn}^2). \quad (129)$$

It is convenient to choose a tetrad comoving with the gas of excitations such that the number of non-zero components of the two free currents is minimized, say $n_p^2 = n_p^3 = n_n^3 = 0$. Hence, let us suppose that an equation of state of the form

$$\mathcal{E} = \mathcal{E}(s^0, n_p^0, n_n^0, n_p^1, n_n^1, n_n^2) \quad (130)$$

is given as the result of a microscopic study of the system. Similarly to the previous example, we perform the change of variables

$$s^0 = \sqrt{s^2} \quad n_p^0 = n_{sp}^2/\sqrt{s^2} \quad n_n^0 = n_{sn}^2/\sqrt{s^2} \quad (131)$$

and

$$n_p^1 = \sqrt{\frac{(n_{sp}^2)^2}{s^2} - n_{pp}^2}. \quad (132)$$

The computation of n_n^1 and n_n^2 requires some extra effort. Since the projector \perp introduced in (83) can be rewritten as

$$\perp^\nu{}_\rho = \delta^\nu{}_\rho + \frac{s^\nu s_\rho}{s^2}, \quad (133)$$

it is immediate to find

$$(\perp n_n)^a = (0, n_n^1, n_n^2, 0). \quad (134)$$

A contraction with $-n_{pa}$ gives

$$n_{pn}^2 - \frac{n_{sn}^2 n_{sp}^2}{s^2} = -n_n^1 n_p^1, \quad (135)$$

which can be cast into the form

$$n_n^1 = \frac{n_{sn}^2 n_{sp}^2 - s^2 n_{pn}^2}{\sqrt{s^2} \sqrt{n_{sp}^4 - n_{pp}^2 s^2}}. \quad (136)$$

Finally, we use the identity

$$n_{nn}^2 = (n_n^0)^2 - (n_n^1)^2 - (n_n^2)^2 \quad (137)$$

to obtain

$$n_n^2 = \sqrt{\frac{n_{sn}^4}{s^2} - \frac{(n_{sn}^2 n_{sp}^2 - s^2 n_{pn}^2)^2}{s^2 (n_{sp}^4 - n_{pp}^2 s^2)}} - n_{nn}^2. \quad (138)$$

We conclude that the covariant hydrodynamic theory for a mixture of two superfluid species generated by the equation of state \mathcal{E} can be constructed starting from the master function

$$\Lambda(s^2, n_{pp}^2, n_{nn}^2, n_{sp}^2, n_{sn}^2, n_{pn}^2) = -\mathcal{E}(s^0, n_p^0, n_n^0, n_p^1, n_n^1, n_n^2), \quad (139)$$

where the arguments of \mathcal{E} are seen as functions of the scalars n_{xy}^2 by means of (131), (132), (136) and (138).

This hydrodynamic model is the basis for many models of neutron star interiors, see e.g. [47]. Even if a proper description should be built starting from (at least) a $(l = 1, k = 2)$ multifluid that includes the normal electrons, the superconducting protons and the superfluid neutrons, see e.g. [48], a two-fluid description is often preferred. For example, [24] consider a Newtonian effective $(l = 0, k = 2)$ uncharged fluid at zero temperature, in which one current is comprised of superfluid neutrons and the other one is a superfluid protons in a neutralizing bath of electrons that plays no dynamical role. The same approach has been used also by [23], where a useful Newtonian formalism to account for entrainment in superfluid neutron stars has been developed. In appendix C we show how to translate the Lagrangian formalism of Prix and collaborators into the Hamiltonian one developed by [24], showing that this operation is more transparent if one considers both the descriptions as low-energy limits of the relativistic multifluid theory.

5. Chemical reactions

The thermodynamic language developed in the previous sections is a useful tool not only for the description of perfect multifluids, but also to discuss some physical situations in which dissipation occurs. In section 5.1 we review the covariant treatment of dissipation that occurs when matter is transformed from one constituent to another due to chemical disequilibrium developed by [49] and applied to beta reactions in neutron star interiors in [18], see also [50] for an analogous formulation in a Milne–Cartan framework. In section 5.2, as a practical application, we invoke the gauge freedom to simplify the rocket term present in the hydrodynamical model of [18] and use the result to prove that in the low temperature limit it reduces to the one considered by [19]. Finally, in section 5.3, we try to rederive the Carter and collaborators’ approach from a purely thermodynamical argument, providing a justification and insight for this treatment of dissipation induced by chemical reactions.

5.1. Chemical reactions in almost-perfect multifluids

Consider the generic perfect multifluid presented in section 4 and imagine to switch on a chemical reaction, defined by (39). If the reaction is slow compared to all the other equilibration processes, such as the friction between the normal components, we can impose that the multifluid is still in local thermodynamic equilibrium with respect to every degree of freedom apart from the chemical fractions [51]. Within this scenario, we can work under the assumption that all the normal currents are still s -locked and that viscous effects are negligible. We refer to this substance as an almost-perfect multifluid: the only form of dissipation is due to chemical reactions, there is no shear viscosity and no heat flow is measured in the frame of the normal components. In other words, the normal components still behave as a mixture that can be modelled as a perfect fluid but the presence of chemical reactions alters the chemical fractions (of both normal and superfluid species) and produces a gain in entropy. In this physical

situation, the energy–momentum tensor keeps the form (17), which implies the following relation

$$\nabla_\rho T^\rho{}_\nu = \sum_x [\mu_\nu^x \nabla_\rho n_x^\rho + 2n_x^\rho \partial_{[\rho} \mu_{\nu]}^x]. \quad (140)$$

Since the equations $\nabla_\rho T^\rho{}_\nu = 0$ and $\partial_{[\rho} \mu_{\nu]}^i = 0$ are still valid and that $n_A^\rho = n_A u^\rho$, we obtain

$$\sum_x \mu_\nu^x \nabla_\rho n_x^\rho + u^\rho \sum_{A=0}^l 2n_A \partial_{[\rho} \mu_{\nu]}^A = 0. \quad (141)$$

A contraction of the above equation with u^ν gives

$$\Theta \nabla_\nu s^\nu = - \sum_{x \neq s} \mu_x^{(T)} \nabla_\nu n_x^\nu. \quad (142)$$

In section 3.3 we introduced the reaction coordinate $N = n^0 V$ as a global quantity which counts the net number of reactions from the left to the right. Now, in a hydrodynamical framework, we need to introduce an associated local analogue. A convenient way to do this is to employ the method of the *virtual charges*: consider (39) and imagine to add a fictitious particle of charge +1 any time a reaction from the left to the right occurs (similarly, a particle of charge −1 is added when the reaction proceeds in the opposite direction). These fictitious particles are just tools to keep track of the reactions: they appear in the point of the space-time where the reaction event happens and, once created, are supposed to be advected by the entropy current.

We define the *reaction current* n^ν to be the particle number current of the +1 charges minus the one of the −1 charges. Since $n^\nu = n u^\nu$, the net density of fictitious charges n coincides with the quantity n^0 that was introduced in section 3.3. This is a useful mathematical construction because, for any $x \neq s$, it allows us to define the conserved currents

$$J_x^\nu := n_x^\nu - (\beta_x - \alpha_x) n^\nu. \quad (143)$$

The conservation laws $\nabla_\nu J_x^\nu = 0$ are just the local differential version of equation (40) and can be used to rewrite (142) as

$$\Theta \nabla_\nu s^\nu = \sum_{x \neq s} (\alpha_x - \beta_x) \mu_x^{(T)} \nabla_\nu n^\nu. \quad (144)$$

The left-hand side describes the production of heat as a result of chemical reactions. The second principle of thermodynamics and the requirement of causality impose that $\nabla_\nu s^\nu \geq 0$, which allows us to write

$$\begin{aligned} \text{if } \sum_{x \neq s} \alpha_x \mu_x^{(T)} > \sum_{x \neq s} \beta_x \mu_x^{(T)} &\implies \nabla_\nu n^\nu \geq 0, \\ \text{if } \sum_{x \neq s} \alpha_x \mu_x^{(T)} < \sum_{x \neq s} \beta_x \mu_x^{(T)} &\implies \nabla_\nu n^\nu \leq 0. \end{aligned} \quad (145)$$

This is just the statement that the reactions proceed from the side having higher chemical potential to the other one: (145) is the local, covariant version of Le Châtelier's principle. Clearly, when chemical equilibrium is reached, see (45), the reactions in the two directions are balanced, implying that there is no net production of fictitious charge,

$$\text{if } \sum_{x \neq s} \alpha_x \mu_x^{(T)} = \sum_{x \neq s} \beta_x \mu_x^{(T)} \implies \nabla_\nu n^\nu = 0. \quad (146)$$

Following [18], we assume that not too far from chemical equilibrium the reaction rates are proportional to the unbalance between the chemical potentials (to ensure the validity of Le Châtelier's principle, the constant of proportionality Ξ must be a positive coefficient). Rewriting (141) using the conservation of J_x^ν we obtain

$$\begin{aligned} \nabla_\nu n^\nu &= \Xi \sum_{x \neq s} (\alpha_x - \beta_x) \mu_x^{(T)} =: r \\ \Theta \nabla_\nu s^\nu &= \Xi \left[\sum_{x \neq s} (\alpha_x - \beta_x) \mu_x^{(T)} \right]^2 \\ \sum_{A=0}^l 2n_A^\rho \partial_{[\rho} \mu_{\nu]}^A &= -\Theta_\nu \nabla_\rho s^\rho + \sum_{x \neq s} (\alpha_x - \beta_x) \mu_\nu^x \nabla_\rho n^\rho. \end{aligned} \quad (147)$$

The right-hand side of the first equation is called *reaction rate* and will be denoted by r in the following. The new quantity r describes the coupling between the hydrodynamic and the chemical evolution of the multifluid.

The presence of superfluid currents has an interesting consequence on the evolution of the system: a relative motion between the species gives rise to an extra term in the hydrodynamic equations, the so-called *rocket effect*, see [52] and [19]. In fact, the third equation of (147) can be seen as an equation of motion for the variable u^ν : the vector

$$\mathcal{R}_\nu := -\Theta_\nu \nabla_\rho s^\rho + r \sum_{x \neq s} (\alpha_x - \beta_x) \mu_\nu^x \quad (148)$$

describes the acceleration of the normal components. This rocket effect results from the fact that the transfusion process produces an overall redistribution of the momenta between the various species. Notice that \mathcal{R}_ν vanishes when all the species comove, as can be seen by using (144) with the assumption that $\Theta_\nu = \Theta u_\nu$ and $\mu_\nu^x = \mu_x^{(T)} u_\nu$.

5.2. How to model chemical reactions in the zero temperature limit

Under a gauge transformation of the type (108), the system of dynamical equation (147) written in terms of the gauge-fixed momenta must be equivalent to the original one. In fact, to derive it we only invoked the the energy–momentum conservation and the irrotationality of the superfluid momenta, which are gauge invariant properties of the system. However, the two sides of the third equation in (147) are not separately gauge invariant: the rocket term transforms as

$$\tilde{\mathcal{R}}_\nu = \mathcal{R}_\nu - \sum_{A=1}^l r_A \pi_\nu^A, \quad (149)$$

with

$$r_A := s^\rho \nabla_\rho \left(\frac{n_A}{s} \right) = (\beta_A - \alpha_A) r - \frac{n_A}{s\Theta} \frac{r^2}{\Xi}. \quad (150)$$

Furthermore, it is possible to check that the left-hand side of the third equation of (147) changes in the same way,

$$\sum_{A=0}^l 2n_A^\rho \partial_{[\rho} \tilde{\mu}_{\nu]}^A = \sum_{A=0}^l 2n_A^\rho \partial_{[\rho} \mu_{\nu]}^A - \sum_{A=1}^l r_A \pi_\nu^A, \quad (151)$$

ensuring that a gauge transformation leaves the overall equation unchanged, as expected.

Realizing that the rocket term $\tilde{\mathcal{R}}_\nu$ is a gauge dependent quantity is particularly important in the modelling of chemical reactions in neutron star interiors in the zero temperature limit, as done e.g. by [19]. Strictly speaking, chemical reactions cannot be consistently introduced in a zero temperature formalism simply because, as it is evident from the second equation in (147), they are associated to an increase of entropy. However, it is possible to perform a gauge fixing in a way that (111) holds; in this way equation (148) becomes

$$\mathcal{R}_\nu = r \sum_{x \neq s} (\alpha_x - \beta_x) \perp_{\nu}^{\sigma} \mu_\sigma^x, \quad (152)$$

where the dependence on $\nabla_\rho s^\rho$ has been eliminated. On the other hand, with this gauge choice we have that $\Theta_\nu = \Theta u_\nu$, which goes to zero in the low temperature limit¹¹. It follows that it is possible to drop the the $A = 0$ term in the sum on the left hand side of (147): this provides a justification of the zero temperature form of the rocket effect used in the analysis of neutron star oscillations made by [19].

Equation (152) is the relativistic version of rocket term derived by [52] in a Newtonian framework (see equation (177) therein) under the assumption of zero entrainment with the entropy. However, we have proven that it is always possible to impose the condition (111) by means of a gauge fixing, so that there is no need to invoke it as a simplifying assumption.

5.3. Thermodynamic approach

In this subsection we give an alternative derivation of the Le Châtelier's principle (145) and of the first two equations in (147) following a purely thermodynamical approach. Our aim is to convince the reader that the common scheme adopted in chemistry to study reactions in a mixture of normal species (see e.g. [53]) is not modified by the presence of currents associated to superfluid species.

Let us consider a box in contact with an ideal heat bath, as described in section 3.1. Assume that the multifluid has been prepared in an homogeneous initial configuration and that the reaction is sufficiently slow compared other equilibration processes. More precisely, the reactions have to be so slow that the evolution of the system is a quasistatic transformation moving across the manifold $\gamma \subset \mathcal{Q}$ of the homogeneous macrostates in which every degree of freedom, apart from the chemical fractions, assumes its equilibrium value. This construction formalizes the concept of quasi-equilibrium described in [54], see also [51].

For simplicity, we consider a single reaction channel, defined by a particular relation of the type (39) and involving an arbitrary number of components. Hence, γ must be a one-dimensional manifold and can be parameterized by a single variable: for this purpose it is

¹¹ With a different gauge fixing the thermal momentum Θ_ν may not go to zero in the low temperature limit. Instead, it may diverge in order to keep the product s^{Θ_j} finite, as can be seen from equation (109) for ζ_ν .

natural to choose n^0 , as it is the parameter that measures the extent in which the reaction proceeds. In particular, remembering that $(n^0, s^0, \Theta_j, \mu_j^x)_{x \in \mathcal{N}}$ is the full set of the $5 + 3l$ independent variables of state that are compatible with the constants of motion, see section 3.3, the curve γ is implicitly defined by the $4 + 3l$ constraints

$$s^j = 0 \quad n_A^j = 0 \quad \Theta = \Theta_H. \quad (153)$$

In principle, these constraints can be used to parametrize Θ_j , μ_j^A and s^0 as functions of n^0 .

Since the system evolves along γ , all we need to know is the set of values $\mathcal{F}(n^0)$ assumed by the free energy on this curve. Combining equations (26), (41) and (153) with the fact that μ_j^i are constants of motion, it is possible to show that the differential of the free energy on γ reads

$$d\mathcal{F} = \sum_{x \neq s} (\beta_x - \alpha_x) \mu_x^{(T)} dn^0, \quad (154)$$

which leads us to introduce the *affinity* of the reaction¹²

$$\mathbb{A} := -\frac{d\mathcal{F}}{dn^0} = \sum_{x \neq s} (\alpha_x - \beta_x) \mu_x^{(T)}. \quad (155)$$

If the minimum of \mathcal{F} does not fall on the border of the domain of n^0 (as discussed in section 3.3), the equilibrium is defined by the minimum of \mathcal{F} over γ , i.e. when $\mathbb{A} = 0$. That is obviously just another way to write equation (45). Reinterpreting (155) in the view of the LAB formalism introduced in section 4.2, it is immediate to show that $\mathbb{A} = 0$ gives exactly the chemical equilibrium condition used by [22] for an electron, proton and neutron mixture, see equation (44) therein.

Now, let us suppose that the initial fractions are slightly out of equilibrium, so that we can expand \mathcal{F} around the equilibrium value of the extent parameter n_{eq}^0 ,

$$\mathcal{F}(n^0) \approx \mathcal{F}_{\text{eq}} + \frac{1}{2} \mathcal{F}_{\text{eq}}'' (n^0 - n_{\text{eq}}^0)^2, \quad \text{with} \quad \mathcal{F}_{\text{eq}}'' := \left. \frac{d^2 \mathcal{F}}{d(n^0)^2} \right|_{n^0 = n_{\text{eq}}^0} > 0. \quad (156)$$

In this case the affinity can be approximated as

$$\mathbb{A} = \mathcal{F}_{\text{eq}}'' (n_{\text{eq}}^0 - n^0). \quad (157)$$

The second principle of thermodynamics forces n^0 to evolve towards the equilibrium value, namely

$$\begin{aligned} \mathbb{A} > 0 &\implies n_{\text{eq}}^0 > n^0 \implies \frac{dn^0}{dt} > 0 \\ \mathbb{A} < 0 &\implies n_{\text{eq}}^0 < n^0 \implies \frac{dn^0}{dt} < 0, \end{aligned} \quad (158)$$

where the precise meaning of the parameter t is provided by the construction in section 3.1. Hence, this result is the homogeneous (purely thermodynamic) version of equation (145).

¹² The affinity is usually defined by means of the Gibbs free energy because in chemistry reactions typically occur at constant pressure. However, in section 3 we assumed a definite control volume V , so we work with the Helmholtz free energy.

¹³ Now that we are dealing with the kinetics of the reaction channel, we need to require that the presence of the heat bath does not alter the speed of reaction. This means that it does not act as a catalyzer nor as an inhibitor.

To advance in the analysis we need to include some kinetic considerations about the speed of the reaction. As we said, the evolution of the system is assumed to be a sequence of states which are in equilibrium with respect to all the variables (but with assigned fractions), so that r can be regarded as a thermodynamic variable itself, and we can write¹³

$$\frac{dn^0}{dt} = r(\gamma, n^0). \quad (159)$$

The dependence on the particular curve γ , defined by the initial state in which the system is prepared, implicitly defines the dependence on the constants of motion that are conserved during the evolution towards equilibrium. Now, if we assume that we are near equilibrium, it is possible use the equation (157) to write r as a function of \mathbb{A} . An expansion of r near $\mathbb{A} = 0$ leads to the formula

$$r \approx \Xi \mathbb{A}, \quad \Xi = \Xi(\gamma) > 0. \quad (160)$$

We have obtained the homogeneous version of the first equation in (147). Notice that this formula is in complete accordance with equations (45) and (50) of [22] and equation (44) of [18], providing a purely thermodynamic justification for both approaches (that may seem different only because these two studies have been developed by using two quite different languages).

To complete the analysis of the hydrodynamical scheme proposed by Carter and collaborators, we have to combine (155), (159) and (160) to find that the heat produced by the chemical reactions contributes to the variation of free energy according to the law

$$\frac{d\mathcal{F}}{dt} = -\Xi \mathbb{A}^2, \quad (161)$$

which is in accordance with the second equation of (147).

There is an interesting final remark to make: in the above analysis we have assumed that the interaction with the heat bath drives the system towards thermal equilibrium with respect to the variables s^0 and μ_j^A faster than with respect to the chemical fractions. This neutralizes the action of the rocket term because, as a result of an ideally instantaneous damping of normal flows in the frame of the bath, the normal species are not allowed to accelerate. For this reason, the third equation in (147) cannot be derived in a purely thermodynamic study: it can arise only in an hydrodynamical treatment. The present analysis of an homogeneous system is, therefore, only complementary to the one of Carter and collaborators. We can exploit the absence of the rocket term in the present treatment to solve exactly equation (159), finding a formula for the time-scale of the reactions, namely

$$\mathbb{A}(t) = \mathbb{A}(0)e^{-t/\tau_{\text{eq}}}, \quad (162)$$

where the equilibration time is

$$\tau_{\text{eq}} = \frac{1}{\Xi \mathcal{F}''_{\text{eq}}}. \quad (163)$$

Therefore, the presence of the rocket term in the hydrodynamic formulation tells us that the dynamics of the reaction and the motion of the components are coupled on the same time-scale τ_{eq} . This leads to the practical complication that these two effects cannot be studied separately unless all the species comove.

6. Conclusions

We have developed a thermodynamic language for a mixture of superfluid and normal species in general relativity, without any approximation regarding the smallness of the relative currents between the components. Our strategy and findings can be summarized as follows:

- (i) The central idea of our analysis has been introduced in section 3.1, where we consider an homogeneous subsystem in contact with an heat bath. We define the total free-energy in terms of three parameters: the temperature of the bath and the entropy and energy of the subsystem, see (21). The link with the phenomenological model of section 2.1 is in the association of the entropy with the density relative to one of the currents (that we called s^μ beforehand) and of the energy with the appropriate component of the energy–momentum tensor, namely equations (22) and (23).
- (ii) Additional links between the state variables of the homogeneous subsystem and the phenomenological model can be provided by considering the quantities conserved during the equilibration process. Of particular interest is the constraint arising from the irrotationality requirement of the superfluid momenta: although the presence of persistent currents breaks the concept of comoving reference frame, we find no difficulties in defining the thermodynamic properties of the system.
- (iii) The absolute temperature and the chemical potentials are defined by considering the relevant physical process (the equilibration with a heat bath in section 3.2 and the relaxation to chemical equilibrium in 3.2).
- (iv) The thermodynamic language developed for the homogeneous system is embedded into the non-homogeneous phenomenological model thanks to the tetrad formalism, leading to the LAB formalism developed by [21] and [10].
- (v) Finally, we described how the phenomenological approach acquires a gauge freedom when its dynamics is restricted to field configurations that are consistent with local equilibrium macrostates. However, the thermodynamic quantities (i.e. the quantities that appear in the LAB formalism) turn out to be gauge invariant. This has interesting consequences on the entrainment matrix and we provided a concrete example of gauge fixing in 4.6 by considering the neutron star crust model of [18].

During our discussion we rediscovered some results already known in the literature, providing further justification that is complementary to the one in the original works. In particular, we derived a chemical equilibrium condition that is consistent with both the one proposed by [18] and the one of [22]. Furthermore, we have shown in section 5.3 that the usual concepts of affinity and reaction coordinates, as they are introduced in chemistry, are not affected by the presence of superfluid currents.

Building on the present formalism, it is possible to extend our work in two main directions.

First, following [46] and [55] it may be interesting to include the effect of quantized vortices providing a description of the kind required for application to relativistic rotating neutron stars. The presence of a vorticity field can have interesting consequences from the thermodynamic point of view, as it is a further degree of freedom that breaks the isotropy of the system and contributes to the energy density of the system. In fact, the procedure to extract the statistical meaning of hydrodynamic quantities defines a rigorous way to study any additional degree of freedom in the equation of state. As we have shown here, in fact, it represents a simple strategy to provide an unambiguous microscopic interpretation to quantities appearing in phenomenological models revealing possible redundant information they may carry.

Secondly, it is tempting to extend the analysis carried out in section 5.1 to study the emergence of reaction-induced viscosity in relativistic models. In upcoming works, the same approach will be applied to the study of dissipation, shown to arise directly from the implementation of out-of-equilibrium degrees of freedom in the equation of state. If the ideas presented in section 5.3 are properly extended, our framework has the potentiality to be applied to a wide range of problems in relativistic astrophysics, ranging from various explosive scenarios to the internal layers of neutron stars.

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Appendix A. Extracting the master function from a RMF model

In section 4.7 we considered two basic examples of how to obtain the master function from a given thermodynamic potential. Here we expand the discussion and sketch a strategy that can be used when the microscopic theory provides directly the energy–momentum tensor. This is the case of the relativistic mean field (RMF) technique, a method used to extract the equation of state starting from the Lagrangian of the quantum fields of interest [36]. In this sense, the RMF is a way to deal with the problem of how to connect the microscopic scale, where one speaks in terms of interacting boson and fermion fields, and the fluid scale, where matter is continuous and described in terms of thermodynamic variables.

The RMF can be used to construct the internal energy of the system when there are no relative currents (this case $\mathcal{U} = -\Lambda = T^{00}$ in the rest frame of matter). If a single relative current between the species is present into the system, obtaining Λ is less trivial but still possible: indeed, [37] generalized the RMF procedure for a mixture of superfluid neutrons and superconducting protons in the outer core of a neutron star starting from a mean field model for the nucleons and their interactions.

In brief, in the RMF the energy–momentum tensor $T^{\nu\rho}$ and the currents n_x^ν of the fluid are calculated as mean field averages of the relative microscopic counterparts. Then, $T^{\nu\rho}$ and n_x^ν have to be combined algebraically in order to extract $\Lambda(n_{xy}^2)$ and the thermodynamic variables of interest. Here, we address this algebraic problem when there are three independent current fields n_1^ν , n_2^ν , n_3^ν , namely two relative currents between the species. This situation

can be obtained in three cases: $(l = 0, k = 3)$ in the zero temperature limit, $(l = 0, k = 2)$ at finite temperature and $(l = 1, k = 2)$ in the zero temperature limit. In the following, the species 2 and 3 are always considered superfluid; the three possibilities are distinguished by the nature of the species 1 which may be a further superfluid, the entropy, or a normal component. Suppose that the current one-forms

$$n_x := n_{x\nu} dx^\nu \quad \text{for} \quad x = 1, 2, 3 \quad (\text{A.1})$$

are linearly independent and define the one-form N as¹⁴

$$N := *(n_1 \wedge n_2 \wedge n_3). \quad (\text{A.2})$$

Using the properties of Hodge duality, it can be proven that

$$N_\nu n_x^\nu = 0 \quad N_\nu N^\nu = \det[n_{xy}^2], \quad (\text{A.3})$$

where $\det[n_{xy}^2]$ is the determinant of the 3×3 symmetric matrix n_{xy}^2 . Since N is orthogonal by construction to every current, we use it to extract the generalised pressure from the energy–momentum tensor:

$$T^{\nu\rho} N_\nu N_\rho = \Psi \det[n_{xy}^2]. \quad (\text{A.4})$$

Taking the trace of the energy–momentum tensor,

$$T^\nu{}_\nu = 3\Psi + \Lambda, \quad (\text{A.5})$$

we obtain that the master function can be constructed as

$$\Lambda(n_{xy}^2) = T^\nu{}_\nu - 3 T^{\nu\rho} \frac{N_\nu N_\rho}{\det[n_{xy}^2]}. \quad (\text{A.6})$$

Thanks to this method we can easily see what happens when the independent currents are more than three (i.e. the system can support more than two relative currents between the components): in this case there are infinite possible algebraic ways of combining the currents to give the same energy–momentum tensor and it is not possible to use them to uniquely reconstruct Λ . The reason is that with four linearly independent currents there is no field N^ν which is orthogonal to every n_x^ν , implying that it is impossible to extract algebraically the generalized pressure by using only the energy–momentum tensor and the currents.

Appendix B. Chemical gauge in an arbitrary perfect multifluid

In this appendix we generalize the idea of chemical gauge introduced in [26] to a generic perfect multifluid. According to the Carter formalism, the fundamental fields used to formulate the hydrodynamic description are the currents n_x^ν . As anticipated in section 2, a chemical species is a subset (labelled by x) of the total amount of particles that, in general, may not reflect the real chemical composition of matter, as was earlier pointed out by [30]. In principle, it is possible to redefine the fundamental currents of the theory using a different prescription to assign each particle to a certain subset x (the entropy has not to be considered a chemical species, so it cannot be redistributed).

¹⁴ The symbol $*$ is the Hodge duality operator: for a generic three-form ω , the duality is defined as $(*\omega)_\mu = \varepsilon^{\nu\rho\sigma} \omega_{\nu\rho\sigma} / 3!$ and $\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g} \text{perm}(\alpha\beta\gamma\delta)$ is the volume four-form. Since $(n_1 \wedge n_2 \wedge n_3)_{\nu\rho\sigma} = 3! n_{[\nu}^1 n_\rho^2 n_{\sigma]}^3$, we have that $N_\mu = \varepsilon_{\nu\rho\sigma\mu} n_1^\nu n_2^\rho n_3^\sigma$. The linear independence of the one-forms n_x ensures that N is not zero.

Imagine now to start with a theory in which the particles have been grouped into $l + k$ subsets. The theory has a chemical gauge freedom if it is possible to redistribute the particles (according to a fixed rule) among the subsets. Clearly, not every redistribution rule is allowable. However, let us start by considering the most general rule,

$$\tilde{n}_y^\nu = \sum_{x \neq s} Z_y^x n_x^\nu, \quad (\text{B.1})$$

where Z is an invertible $(k + l) \times (k + l)$ matrix whose components are functions of all the scalars n_{xy}^2 . The possibility to perform such a transformation simply reflects the freedom to choose the coordinates of the master function, so it does not alter the equations of motion. The matrix Z_y^x , however, is not completely arbitrary because of the further (physically motivated) subdivision of the currents into s -locked and free: the redefined currents \tilde{n}_A^ν still have to behave like normal currents, while the new momenta $\tilde{\mu}_\nu^i$ must be related to their order parameters. To implement the fact that the subdivision into s -locked and free currents should be preserved, we have to enforce that

$$\tilde{n}_A^\nu = \tilde{n}_A u^\nu \quad \frac{\partial \Lambda}{\partial \tilde{n}_i^\nu} = \frac{\partial \Lambda}{\partial n_i^\nu}. \quad (\text{B.2})$$

The first condition implies $Z_A^i = 0$. The second one is more subtle and explicitly reads

$$\sum_{x \neq s} (Z^{-1})^i_x \mu_\nu^x + \sum_{x, y \neq s} \mu_\rho^x \tilde{n}_y^\rho \frac{\partial (Z^{-1})^y_x}{\partial \tilde{n}_i^\nu} = \mu_\nu^i. \quad (\text{B.3})$$

The simplest way to satisfy the above equation is to require $Z_h^i = \delta_h^i$ and that the full matrix Z does not depend on the superfluid currents. Hence, equation (B.1) reduces to

$$\tilde{n}_A^\nu = \sum_{B=1}^l Z_A^B n_B^\nu \quad \tilde{n}_i^\nu = \sum_{B=1}^l Z_i^B n_B^\nu + n_i^\nu, \quad (\text{B.4})$$

with $Z_x^A = Z_x^A(n_B, s)$. There is a further constraint which is worth noticing, related to the conservation of the currents. Suppose that there is a conserved current J^ν and that each particle of the species x carries a formal charge q^x . This gives the constraint

$$J^\nu = \sum_{x \neq s} q^x n_x^\nu = \sum_{x \neq s} q^x \tilde{n}_x^\nu. \quad (\text{B.5})$$

A comparison with (B.1) tells us that $\sum_y Z_y^x q^y = q^x$. However, considering that $Z_x^i = \delta_x^i$, we see that the only non-trivial condition implied by (B.5) is

$$\sum_{x \neq s} Z_x^A q^x = q^A. \quad (\text{B.6})$$

Furthermore, to have complete chemical gauge invariance Z_x^A should be constant along the worldlines: only in this way $\nabla_\nu \tilde{n}_x^\nu$ vanishes, giving rise to a Lagrangian description which is completely equivalent to the one obtained using the n_x^ν . In particular, the energy–momentum tensor, as defined by (17) is invariant with respect to such a change of chemical basis.

To provide an explicit example, we show how the transformation (B.4) comes down to the chemical gauge transformation considered in [26], see equation (2.23) therein. Carter and collaborators considered three species in the crust of a neutron star, two s -locked (the ions that

constitute the crustal lattice, I , and the protons, p) and one free (the neutrons, n) and perform the change of variables,

$$\tilde{n}_I^\nu = n_I^\nu \quad \tilde{n}_p^\nu = a_c n_p^\nu \quad \tilde{n}_n^\nu = (1 - a_c) n_p^\nu + n_n^\nu, \quad (\text{B.7})$$

where $a_c = a_c(n_I, n_p)$ and the conserved current J^ν is the total baryon current $n_b^\nu = n_p^\nu + n_n^\nu$ (namely $q^p = q^n = 1, q^I = 0$). Clearly, this transformation has the general form (B.4), where Z must fulfill also the additional constraint (B.5).

Appendix C. An example of Hamiltonian formulation in the Newtonian limit

In this appendix we address the practical problem of how to translate two different Newtonian formulations, the one of [23] and the one of [24], for the hydrodynamics of the outer core of a neutron star under conditions when both neutrons (n) and protons (p) are superfluid. Since the model has been formulated for matter at zero temperature, there is no entropy current, i.e. both formulations consider two superfluid species at zero temperature (i.e. $k = 2, l = 0$). Our aim is to provide a clear example of how a fully covariant theory may be used to build a bridge between two Newtonian theories that are apparently different. Moreover, the construction of a clear dictionary between these two different approaches has its practical usefulness for the reader interested in the modelling of neutron star interiors.

C.1. The Newtonian limit of the Carter formalism

Let us briefly review, following [52], how to obtain the Newtonian limit of the Carter formalism¹⁵. To make the discussion more concrete, we stick to the case ($k = 2, l = 0$) for two superfluid species at zero temperature. In fact, the zero temperature case with two independent currents is of practical interest as it provides the basic framework to model neutron star interiors (see e.g. [7–9, 12, 38]). We start by writing the potential \mathcal{E} as a function of the rest frame densities of protons n_p and neutrons n_n and of the relative speed Δ ,

$$d\mathcal{E} = \mu_p^{(C)} dn_p + \mu_n^{(C)} dn_n + \alpha d\Delta^2, \quad (\text{C.1})$$

where $\mu_x^{(C)}$ are the comoving chemical potentials defined in 64. It is easy to show that the quantity α is related to the anomaly coefficient via

$$\alpha = \frac{1}{2} \mathcal{A} n_{np}^2 \Gamma^2 = \frac{1}{2} \mathcal{A} n_n n_p \Gamma^4, \quad (\text{C.2})$$

where $\Gamma = n_{np}^2 = (1 - \Delta^2)^{-1/2}$ is the Lorentz factor associated to the relative motion between the two species. With the perspective of taking the Newtonian limit we write \mathcal{E} as

$$\mathcal{E} = mn_p + mn_n + \tilde{\mathcal{E}}(n_p, n_n, \Delta^2), \quad (\text{C.3})$$

where m is the mass per particle of the mixture (this definition of m allows us to forget about the electrons that, in principle, should be present into the system and define a third fluid). The differential of $\tilde{\mathcal{E}}$ is, then,

¹⁵ Strictly speaking the formalism of [52], which generalizes the model of [23], is the Newtonian counterpart of the relativistic hydrodynamical description of superfluid neutron stars of [38]. The difference with respect to the language adopted by Carter is mainly in the definition of the entrainment parameters: the approaches of Carter and Prix are completely equivalent and it is possible to translate one into the other.

$$d\tilde{\mathcal{E}} = \mu_p^{ACP} dn_p + \mu_n^{ACP} dn_n + \alpha d\Delta^2, \quad (\text{C.4})$$

where

$$\mu_x^{ACP} = \mu_x^{(C)} - m \quad (\text{C.5})$$

is the non-relativistic version of the comoving chemical potential.

Following the notation used by [24], we denote with μ_x^{ACP} the chemical potential used in [56], as well as in [52]. Considering that

$$n_x = \sqrt{(n_x^0)^2 - \mathbf{n}_x^2} \approx n_x^0 - \frac{1}{2} \frac{\mathbf{n}_x^2}{n_x^0}, \quad (\text{C.6})$$

the total master function reads

$$\Lambda \approx -mn_p^0 - mn_n^0 + \frac{m\mathbf{n}_p^2}{2n_p^0} + \frac{m\mathbf{n}_n^2}{2n_n^0} - \tilde{\mathcal{E}}(n_p^0, n_n^0, \Delta^2). \quad (\text{C.7})$$

The Newtonian master function is defined as the relativistic one deprived of the rest-mass energy density contribution, namely

$$\tilde{\Lambda} := \Lambda + mn_p^0 + mn_n^0 = \frac{m\mathbf{n}_p^2}{2n_p^0} + \frac{m\mathbf{n}_n^2}{2n_n^0} - \tilde{\mathcal{E}}. \quad (\text{C.8})$$

Now, following definition (16),

$$\begin{aligned} \mu_j^p &= \frac{\partial \Lambda}{\partial n_p^j} = \frac{\partial \tilde{\Lambda}}{\partial n_p^j} = mv_{pj} - \frac{2\alpha}{n_p^0} (v_{pj} - v_{nj}) \\ \mu_j^n &= \frac{\partial \Lambda}{\partial n_n^j} = \frac{\partial \tilde{\Lambda}}{\partial n_n^j} = mv_{nj} - \frac{2\alpha}{n_n^0} (v_{nj} - v_{pj}), \end{aligned} \quad (\text{C.9})$$

where

$$v_x^j = \frac{n_x^j}{n_x^0} \quad (\text{C.10})$$

is the velocity of the species x . For later convenience we introduce also the symbol

$$\epsilon_x := \frac{2\alpha}{mn_x} \quad (\text{C.11})$$

which allows us to rewrite (C.9) in the form

$$\mu_j^x = m[(1 - \epsilon_x)v_{xj} + \epsilon_x v_{yj}] \quad x \neq y. \quad (\text{C.12})$$

The entrainment parameters α and ϵ_x have, in general, an explicit dependence on the relative speed Δ . However, in [24] the entrainment coefficients are supposed, coherently with a low velocity limit, to be functions only of the densities, so that we should perform a first order expansion around $\Delta^2 = 0$, namely (see C.4)

$$\tilde{\mathcal{E}}(n_p^0, n_n^0, \Delta^2) \approx \tilde{\mathcal{U}}(n_p^0, n_n^0) + \alpha(n_p^0, n_n^0) \Delta^2, \quad (\text{C.13})$$

¹⁶ We recall that if all the species comove \mathcal{E} coincides with \mathcal{U} , see (57), so $\tilde{\mathcal{E}}(n_p^0, n_n^0, \Delta^2 = 0) = \tilde{\mathcal{U}}(n_p^0, n_n^0)$.

where the quantity $\tilde{\mathcal{U}}$ is the non-relativistic energy density for comoving species as measured in their common frame¹⁶.

C.2. Introducing the Hamiltonian

As a first step, we introduce the concept of Hamiltonian for a two-fluid system (the multifluid generalization should be obvious). This is done by considering the energy–momentum tensor in a certain frame but, since in the model of Kobayakov and Pethick there is no entropy current, we are forced to use an observer \mathcal{O} that is completely generic. The energy density measured in the frame \mathcal{O} can be obtained as

$$\mathcal{H}_{\mathcal{O}} = T^{00} = -\Lambda + n_p^j \mu_j^p + n_n^j \mu_j^n, \quad (\text{C.14})$$

which is the Legendre transform of Λ with respect to n_x^j . Its differential reads

$$d\mathcal{H}_{\mathcal{O}} = \mu^{p0} dn_p^0 + \mu^{n0} dn_n^0 + n_p^j d\mu_j^p + n_n^j d\mu_j^n. \quad (\text{C.15})$$

Again, before taking the Newtonian limit we subtract the rest mass, defining the Hamiltonian density

$$\tilde{\mathcal{H}}_{\mathcal{O}} := \mathcal{H}_{\mathcal{O}} - mn_p^0 - mn_n^0. \quad (\text{C.16})$$

Taking the differential of both sides it is evident that

$$d\tilde{\mathcal{H}}_{\mathcal{O}} = \mu_p^{\mathcal{O}} dn_p^0 + \mu_n^{\mathcal{O}} dn_n^0 + n_p^j d\mu_j^p + n_n^j d\mu_j^n, \quad (\text{C.17})$$

where

$$\tilde{\mu}_p^{\mathcal{O}} = \mu^{p0} - m \quad \tilde{\mu}_n^{\mathcal{O}} = \mu^{n0} - m, \quad (\text{C.18})$$

are the non-relativistic chemical potentials in the frame of \mathcal{O} , not to be confused with the thermodynamic (44) or the comoving (64) ones. Finally,

$$\tilde{\mathcal{H}}_{\mathcal{O}} = -\tilde{\Lambda} + n_p^j \mu_j^p + n_n^j \mu_j^n \quad (\text{C.19})$$

is the analogous of (C.14) in the Newtonian limit.

C.3. Connection between the Lagrangian and Hamiltonian approaches

Using the definition of the momenta in (C.9), it is possible to prove the useful identity,

$$\frac{1}{2} n_p^j \mu_j^p + \frac{1}{2} n_n^j \mu_j^n = \frac{m n_p^2}{2 n_p^0} + \frac{m n_n^2}{2 n_n^0} - \alpha \Delta^2, \quad (\text{C.20})$$

that, used with (C.8) and (C.13), allow to cast the energy density given in (C.19) into the form

$$\tilde{\mathcal{H}}_{\mathcal{O}} = \tilde{\mathcal{U}} + \frac{1}{2} n_p^j \mu_j^p + \frac{1}{2} n_n^j \mu_j^n. \quad (\text{C.21})$$

This expression highlights the separation into a kinetic and an internal part, see with equation (10) in [24]. This separation leads naturally to define a new chemical potential for each species as

$$\mu_x^{KP} := \frac{\partial \tilde{\mathcal{U}}}{\partial n_x^0} \quad \text{for } x = n, p, \quad (\text{C.22})$$

which coincides with the thermodynamic chemical potential $\mu_x^{(T)}$ in the limiting case in which all species comove. The label *KP* is used to stress that this is the notion of chemical potential used by [24].

To complete the dictionary we need to relate the two formulations of entrainment: in [52] the notation (66) is used, while (69) is preferred in [24]. Inverting (C.12), it is immediate to obtain

$$n_x^j = \frac{n_{xx}^{KP}}{m} \mu^{xj} + \frac{n_{xy}^{KP}}{m} \mu^{yj} \quad \text{with } x, y = \{n, p\}, x \neq y, \quad (\text{C.23})$$

where the diagonal element of the n_{xy}^{KP} entrainment matrix are

$$n_{pp}^{KP} = \frac{n_p^0(1 - \epsilon_n)}{1 - \epsilon_p - \epsilon_n} \quad n_{nn}^{KP} = \frac{n_n^0(1 - \epsilon_p)}{1 - \epsilon_n - \epsilon_p}, \quad (\text{C.24})$$

while the off-diagonal ones are

$$n_{pn}^{KP} = -\frac{n_p^0 \epsilon_p}{1 - \epsilon_p - \epsilon_n} \quad n_{np}^{KP} = -\frac{n_n^0 \epsilon_n}{1 - \epsilon_p - \epsilon_n}. \quad (\text{C.25})$$

Here, the coefficients n_{xy}^{KP} do not depend on the relative speed and obey, as it should be, to the symmetry condition $n_{np}^{KP} = n_{pn}^{KP}$. They, also satisfy, considering (C.11), the equations

$$n_{pp}^{KP} + n_{pn}^{KP} = n_p^0 \quad n_{nn}^{KP} + n_{np}^{KP} = n_n^0,$$

which can be derived directly from the Newtonian limit of the relativistic equation $T^{0j} = T^{j0}$, namely

$$n_p^0 \mu_j^p + n_n^0 \mu_j^n = m(n_{pj} + n_{nj}). \quad (\text{C.26})$$

Now that we know how to translate the entrainment coefficients, we can come back to the Hamiltonian for the system and rewrite (C.21) in the form

$$\tilde{\mathcal{H}}_{\mathcal{O}} = \tilde{\mathcal{U}}(n_p^0, n_n^0) + \frac{n_p^0}{2m} \boldsymbol{\mu}_p^2 + \frac{n_n^0}{2m} \boldsymbol{\mu}_n^2 - \frac{n_{np}^{KP}}{2m} |\boldsymbol{\mu}_p - \boldsymbol{\mu}_n|^2, \quad (\text{C.27})$$

which, remembering equation (C.17), implies

$$\tilde{\mu}_x^{\mathcal{O}} = \mu_x^{KP} + \frac{\boldsymbol{\mu}_x^2}{2m} - \frac{\partial n_{np}^{KP}}{\partial n_x^0} \frac{|\boldsymbol{\mu}_p - \boldsymbol{\mu}_n|^2}{2m}. \quad (\text{C.28})$$

This relation allows to link the notion of chemical potential used by Kobayakov and Pethick to the chemical potentials in (C.18).

C.4. The problem of the chemical potentials

In [57], the Euler-like equations for the two-fluid system derived in [24] are compared with those expected in the context of Prix's theory. Kobayakov and Pethick finds out that they coincide if it is true that

$$\mu_x^{ACP} - \frac{1}{2} m \epsilon_x^2 \Delta^2 = \mu_x^{KP} - \frac{\partial n_{np}^{KP}}{\partial n_x^0} \frac{|\boldsymbol{\mu}_p - \boldsymbol{\mu}_n|^2}{2m}. \quad (\text{C.29})$$

This is a non-immediate translation problem whose solution can be conveniently found by considering that the two formulations are the Newtonian limit of the same relativistic theory.

Let us use equations (C.5) and (C.18) into the definition of the comoving chemical potentials (64):

$$m_x + \mu_x^{ACP} = (m_x + \tilde{\mu}_x^{\mathcal{O}}) u_x^0 - \mu_{xj} u_x^j. \quad (\text{C.30})$$

The Newtonian limit of the above equation is immediately obtained thanks to the fact

$$u_x^0 \approx 1 + \frac{1}{2} \mathbf{v}_x^2 \quad u_x^j \approx v_x^j, \quad (\text{C.31})$$

so that, neglecting higher orders and imposing the approximation $m_x \approx m$, equation (C.30) reduces to

$$\mu_x^{ACP} = \tilde{\mu}_x^{\mathcal{O}} + \frac{1}{2} m \mathbf{v}_x^2 - \mu_{xj} v_x^j. \quad (\text{C.32})$$

Employing equations (C.28) and (C.12), we finally arrive at

$$\begin{aligned} \mu_x^{ACP} &= \mu_x^{KP} - \frac{\partial n_{np}^{KP}}{\partial n_x^0} \frac{|\boldsymbol{\mu}_p - \boldsymbol{\mu}_n|^2}{2m} + \frac{\mu_x^2}{2m} + \frac{m \mathbf{v}_x^2}{2} - \mu_{xj} v_x^j \\ &= \mu_x^{KP} - \frac{\partial n_{np}^{KP}}{\partial n_x^0} \frac{|\boldsymbol{\mu}_p - \boldsymbol{\mu}_n|^2}{2m} + \frac{1}{2} m \epsilon_x^2 \Delta^2, \end{aligned} \quad (\text{C.33})$$

which is what we wanted to prove.

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Coauthor Statement

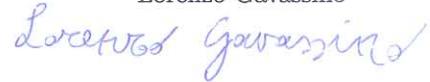
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Declaration of Lorenzo Gavassino (first author):

I, Lorenzo Gavassino, hereby declare that my contribution to the aforementioned research article amounts to 75%.
I performed the mathematical analysis entirely, mastering all the techniques and formalisms that appear in the paper.
I also wrote the draft.

Lorenzo Gavassino



Declaration of Marco Antonelli (second author):

I, Marco Antonelli, hereby declare that my contribution to the aforementioned research article amounts to 25%.
I introduced Lorenzo Gavassino (the first author) to the problem, engaging him in conversations which led to the formation of the paper. I also contributed to the preparation of the draft, improving the exposition.

Marco Antonelli



Chapter 5

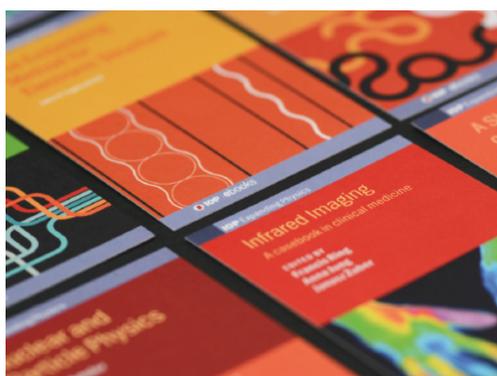
**Bulk viscosity in relativistic fluids:
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PAPER

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Bulk viscosity in relativistic fluids: from thermodynamics to hydrodynamics

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Abstract

The approach of extended irreversible thermodynamics consists of promoting the dissipative fluxes to non-equilibrium thermodynamic variables. In a relativistic context, this naturally leads to the formulation of the theory of Israel and Stewart (1979), which is, to date, one of the most successful theories for relativistic dissipation. Although the generality of the principle makes it applicable to any dissipative fluid, a connection of the Israel–Stewart theory with microphysics has been established, through kinetic theory, only for the case of ideal quantum gases. By performing a convenient change of variables, we provide, for the case of bulk viscosity, an equivalent reformulation of the equations at the basis of extended irreversible thermodynamics. This approach maps any thermodynamic process which contributes to the bulk viscosity into a set of chemical reactions, whose reaction coordinates are abstract parameters describing the displacement from local thermodynamic equilibrium of the fluid element. We apply our new formalism to the case of the relativistic fluids, showing that the Israel–Stewart model for bulk viscosity is just the second-order expansion of a minimal model belonging to a larger class of non-perturbative theories for bulk viscosity which include the nuclear-reaction-mediated bulk viscosity of neutron star matter as a particular case. Furthermore, we show with concrete examples that our formalism provides new ways of computing the bulk viscosity directly and defines a simple prescription for constructing the Israel–Stewart model for a generic bulk-viscous fluid.

Keywords: relativistic hydrodynamics, dissipative fluids, bulk viscosity, extended irreversible thermodynamics

(Some figures may appear in colour only in the online journal)

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1. Introduction

The recent detection of gravitational waves from a neutron star binary inspiral, event GW170817 [1], together with the full range of electromagnetic emission in the following hours to months [2], has allowed for an unprecedented insight into the physics of hot dense matter [3]. Future observations are likely to provide more valuable information on physics in such extreme conditions, and numerical relativity simulations will play a key role in interpreting the data.

Most simulations to date, with a few notable exceptions [4–7], have not included the transport properties of the matter, in the assumption that their effect is negligible on the dynamical time-scales of interest. The remnant of a neutron star merger is, however, a hot, metastable, neutron star surrounded by a thick torus, and the evolution of the system, including the ejecta and associated electromagnetic emission, will be strongly influenced by viscosity [8, 9]. In particular the interior of the merger remnant is characterised not only by high densities but also temperatures above 10 MeV [10], and in these conditions the transport time-scales are comparable to the dynamical time-scales of the system, and thermal transport and bulk viscosity play a key role in the evolution [11].

Besides its importance for applications to simulations of neutron star mergers, bulk viscosity has been shown to be a distinctive feature of fluids coupled with radiation [12, 13]. It also plays an important role in heavy ion collisions [14] and it is the only relevant dissipative process in homogeneous cosmologies [15].

In the standard Navier–Stokes formulation, the dissipative terms of the energy–momentum tensor are taken to be proportional to the spatial derivatives of the fundamental hydrodynamic fields, temperature (for the heat flux) and velocity (for viscosity). This approach, which leads to a parabolic system [16], has been shown to lead to disastrous consequences in a relativistic framework, producing unstable [17], and non-causal solutions. Recent studies have also shown that the entropy of these fluids is not maximum in equilibrium [18], which makes the theory inconsistent with the principles of thermodynamics and constitutes the deep origin of the instability.

More successful attempts to model dissipation in a relativistic framework have been carried out by Stewart [19] and Israel and Stewart [20]. Following an approach which was later systematized by Jou *et al* [21], known as *extended irreversible thermodynamics*, the dissipative fluxes (e.g. viscous stresses and heat flux) are treated as further variables in the equation of state, which parametrise the displacement from equilibrium of the fluid elements. In this way one is naturally led to write telegraph-type equations which, besides a source term containing the spatial derivatives, include a relaxation term. This encodes the delay of response of the fluid element to any displacement from equilibrium produced by the hydrodynamic motion and contains the information about the time-scale needed to the relaxation processes to restore local thermodynamic equilibrium.

Despite the many successes of this approach, which has been proven to be causal and stable in the limit of small deviations from equilibrium [22], still some problems remain. It has been shown to be non-causal and unstable when the deviations from equilibrium become large [23]. This is not so surprising, considering that the formalism is a perturbative expansion near equilibrium and is not expected to hold in a more general context. On the other hand, for a philosophical desire of completeness of the theory, as well as for a more practical requirement of reliability of the numerical implementation, one would like to have a theory able, at least in principle, to deal with an arbitrary large displacement from equilibrium (provided that a hydrodynamic description remains possible). Furthermore, the contact of the Israel–Stewart theory with microscopic models has been clearly established only for ideal quantum gases, for which

a full kinetic description is available. The application of the model to liquids or multifluids still constitutes a challenge [24].

Bulk viscosity is the result of the competition between the dissipative processes which try to maintain the fluid elements in local thermodynamic equilibrium and the hydrodynamic expansion of their volume, which drives them out of it. In this paper we show that, if one assumes the validity of the principles of extended irreversible thermodynamics (which we summarise in subsection 2.1) and performs a convenient change of variables, any source for bulk viscosity can be modelled as a set of effective chemical reactions. This enables us to transform any Israel–Stewart [20] bulk-viscous fluid into an effective multi-constituent single fluid.

The mathematical structure of our approach ultimately builds on the multifluid hydrodynamical formulation of Carter and collaborators [25–28], by supplementing it with an equation of state that depends on entropy density (which allows for a causal description of heat transport, as shown e.g. in [29–31]), and the ‘slow’ degrees of freedom which give rise to bulk viscosity during an expansion. This naturally provides a general relativistic formulation for the viscous hydrodynamics which is symmetric hyperbolic and causal [32], and has a clear link to microphysics and kinetic theory (see also [33] for an alternative discussion of the connection between microphysical quantities and hydrodynamics in a relativistic context).

We also provide two concrete examples, respectively for the case of bulk viscosity in neutron stars induced by β reactions and for the case of an ideal gas of neutrons, and compare our approach to that of Israel and Stewart [20]. In the first case we derive the formulas for the coefficients of Israel–Stewart theory (which in our formalism is simply the expansion near thermodynamic equilibrium of a more general model) in terms of thermodynamic quantities appearing in the two-component model. In the second case we find that for temperatures below 1 MeV ($\approx 10^{10}$ K) our model, which builds directly on the evolution of the momentum distributions of the particles, converges to [20]. However at higher temperatures, such as those of a neutron star merger remnant, the results differ.

Throughout the paper we adopt the spacetime signature $(-, +, +, +)$ and work in natural units $c = G = k_B = 1$.

2. Thermodynamics of out-of-equilibrium fluids

In this section we extend the thermodynamic formalism to substances which are not in equilibrium. In general, an out-of-equilibrium system may be impossible to study without taking into account all the microscopic degrees of freedom: in the absence of an equation of state involving a limited set of macroscopic variables any hydrodynamic description of the system would be incomplete and one should rely on kinetic theory. Such a system would be beyond the present discussion. Instead, coherently with the assumptions of extended irreversible thermodynamics [21], we analyse systems in which it is possible to identify a limited number of macroscopic degrees of freedom containing all the information we need. We show that, if the volume element is isotropic, there is a thermodynamic equivalence with reacting mixtures.

2.1. Quasi-equilibrium states

Consider a macroscopic portion of a fluid, comprised of N particles, enclosed in a cubic box of volume $V = L^3$, surrounded by perfectly reflecting walls. We impose that N is fixed (we are assuming that N is a conserved charge of the underlying microscopic theory, e.g. baryon number), but allow for external variations of V . The system has been prepared at rest in a configuration that is homogeneous and this property will be preserved during the whole evolution.

Therefore, the density of particles n and the energy density \mathcal{U} are everywhere given by

$$n = \frac{N}{V} \quad \mathcal{U} = \frac{E}{V}, \quad (1)$$

where E is the total mass–energy of the fluid. Since the number of particles is fixed, it is more convenient to consider, as fundamental intensive variables, the volume and the energy per particle,

$$v = \frac{V}{N} = \frac{1}{n} \quad \tilde{\mathcal{U}} = \frac{E}{N} = \frac{\mathcal{U}}{n}. \quad (2)$$

If the walls are fixed (so the volume is conserved) and adiabatic (so the energy is conserved), the system is isolated. In this case the walls play the role of external fields and appear in the microscopic Hamiltonian as parameters, not as additional degrees of freedom [34]. Assuming that there are no other constants of motion, after an equilibration process the fluid will reach a homogeneous equilibrium macrostate whose properties will be functions only of V , N and E .

Let us assume that the dynamics of the equilibration shows two different time-scales. In particular we impose that there is a limited set of independent observables α_A , $A = 1, \dots, l-1$ (the reason why we denote the amount of variables as $l-1$ will be clear soon) describing homogeneous local properties of the fluid elements, whose equilibration time τ_M is large with respect to the equilibration time τ_m of all the remaining microscopic degrees of freedom. This allows us to define a manifold \mathcal{Z} of the quasi-equilibrium¹ macrostates

$$(v, \tilde{\mathcal{U}}, \alpha_1, \dots, \alpha_{l-1}). \quad (3)$$

Since our aim is to encode only bulk viscosity effects, neglecting shear viscosity and heat flow, we impose that any anisotropy equilibrates in the time-scale τ_m , therefore the quasi-equilibrium states of matter elements are all invariant under rotation (i.e. isotropic).

We can introduce the entropy of the quasi-equilibrium macrostates S . Its amount per particle is

$$x_s = \frac{S}{N} \quad (4)$$

and is a function of the variables of state given in (3). Since there are no other constants of motion, the thermodynamic equilibrium macrostate of the fluid, which is reached in a time τ_M , maximizes x_s compatibly with the constraints $\delta N = 0$, $\delta V = 0$ and $\delta E = 0$, giving

$$\left. \frac{\partial x_s}{\partial \alpha_A} \right|_{v, \tilde{\mathcal{U}}, \alpha_B} = 0, \quad \forall A, B \neq A. \quad (5)$$

The usual approach of extended irreversible thermodynamics presented by Jou *et al* [21] consists of choosing the bulk viscosity Π as the unique non-equilibrium state variable α_A , producing a model with $l = 2$. This choice would lead us directly to a rediscovery of the Israel–Stewart theory for bulk viscosity [20]. In this paper, however, we propose a more universal approach, which can be applied also to cases with $l > 2$. A simple example of a system with $l = 3$ is

¹The term quasi-equilibrium should not be confused with near-equilibrium. The first term means that all the degrees of freedom apart from a limited amount of macroscopic variables is in equilibrium. The second one means that all the degrees of freedom assume an average value which is near to the equilibrium one. A quasi-equilibrium state is also near-equilibrium when the deviation from equilibrium of the variables α_A is small. This is not required in the present discussion.

radiation hydrodynamics, in those cases in which both the photon number and the average photon energy behave as two independent (non-conserved) dynamical degrees of freedom [13]. Another context in which a model with $l = 3$ might be unavoidable is the hydrodynamic modelling of the bulk viscosity in holographic strongly coupled gauge theories [35].

2.2. The chemical-like chart

To embed the thermodynamic discussion into a hydrodynamic model we can construct a convenient global chart on \mathcal{Z} .

Suppose the system has been prepared in an initial state belonging to \mathcal{Z} and to move the walls of the box to produce a variation δV of the volume in a time τ_{fr} such that

$$\tau_{\text{m}} \ll \tau_{\text{fr}} \ll \tau_{\text{M}}. \quad (6)$$

Since this transformation is slow, the microscopic degrees of freedom have time to equilibrate instantaneously during the process (i.e. the system moves along a curve of quasi-equilibrium states in \mathcal{Z}). This implies that there is no entropy production due to the movement of the walls [34]. On the other hand, the transformation is fast with respect to τ_{M} , implying that the equilibration processes arising from the fact that the α_A can be out of equilibrium do not have time to occur. This transformation is adiabatic and reversible, namely

$$\delta S = 0. \quad (7)$$

Given an arbitrary point in \mathcal{Z} we are able to draw a curve which crosses it (parametrised with the volume per particle v) that describes the states which can be reached starting from the point and making an expansion, or contraction, of the volume in the time-scale τ_{fr} . Since the process is reversible, this curve is unique and is the same if we take any point belonging to it as the starting point. We can, thus, define a vector field W_{fr} on \mathcal{Z} to be the generator of the flux whose orbits are these curves. It satisfies

$$W_{\text{fr}}(v) = 1 \quad W_{\text{fr}}(x_s) = 0, \quad (8)$$

where we have also used the fact that the number of particles in the box is conserved during an adiabatic expansion. It is always possible (see appendix A1) to find a global chart of \mathcal{Z} such that the components of W_{fr} assume the form of a Kronecker delta². It is evident from (8) that in such a chart one of the coordinates can be taken to be v , giving

$$W_{\text{fr}} = \frac{\partial}{\partial v}, \quad (9)$$

and an other coordinate can be x_s . We construct the remaining coordinates to be dimensionless and denote them x_A for $A = 1, \dots, l-1$. Given the fact that in this chart equation (9) holds, we have that $W_{\text{fr}}(x_A) = 0$. Hence, whatever the new internal degrees of freedom are, it is always possible to choose the $l-1$ additional variables that are conserved in fast adiabatic expansions.

Writing the energy per particle as a function of these state variables, its differential on \mathcal{Z} reads

$$d\tilde{u} = \Theta dx_s + \mathcal{K} dv - \sum_{A=1}^{l-1} \mathbb{A}^A dx_A, \quad (10)$$

² Given a smooth non-vanishing vector field on a manifold, this ‘straightfication’ can always be done locally [36]. In appendix A1 we prove that in our case a global construction is also possible.

where the symbols Θ and \mathbb{A}^A bear an analogy with equilibrium thermodynamics [37]. In particular, Θ represents the quasi-equilibrium generalization of the notion of temperature. Isolating dx_s we find

$$dx_s = \frac{1}{\Theta} d\tilde{\mathcal{U}} - \frac{\mathcal{K}}{\Theta} dv + \sum_{A=1}^{l-1} \frac{\mathbb{A}^A}{\Theta} dx_A. \quad (11)$$

Comparing with (5), we find that when the thermodynamic equilibrium is reached

$$\mathbb{A}^A = 0 \quad \forall A = 1, \dots, l-1. \quad (12)$$

Therefore, we call the quantities \mathbb{A}^A *generalised affinities*.

In fact, the coordinates x_A are analogous to chemical fractions (or, better, reaction coordinates). Consider a reacting multicomponent system in which the equilibration processes of the momenta of the particles are much faster than the reaction rates. Then, make an expansion which is sufficiently slow that in any instant the momenta of the particles are in their equilibrium distribution, but sufficiently fast that no reactions have time to occur. In this sense chemical fractions can be regarded as the archetype of an internal degree of freedom, to be included into the equation of state, which behaves as a frozen variable under sufficiently fast volume expansions (see e.g. [38]). However, we have proven that these state variables can be constructed in an arbitrary system (even in the absence of real chemical reactions).

For example, we will explicitly consider the case of a simple gas, where there are no chemical reactions: we will identify some variables x_A of this kind and use them to build a chart over the quasi-equilibrium states (this is shown in section 8 starting from the kinetic description of a simple gas).

2.3. Effective multi-constituent equation of state

If we define the quantities

$$s = nx_s \quad n_A = nx_A, \quad (13)$$

where s is the entropy per unit volume, equations (2) and (10) give

$$d\mathcal{U} = \Theta ds + \mu dn - \sum_{A=1}^{l-1} \mathbb{A}^A dn_A. \quad (14)$$

Here, μ is the generalization of the chemical potential to quasi equilibrium states, satisfying the condition

$$\mathcal{K} = \mathcal{U} - \Theta s - \mu n + \sum_{A=1}^{l-1} \mathbb{A}^A n_A. \quad (15)$$

The above relation, a *generalised Euler relation*, defines a Legendre transformation, implying that

$$d\mathcal{K} = -s d\Theta - n d\mu + \sum_{A=1}^{l-1} n_A d\mathbb{A}^A. \quad (16)$$

The variable \mathcal{K} is, therefore, the *generalised grand potential density*, as it would be if the quantities n_A were interpreted as densities of chemical species and the affinities as the respective

chemical potentials. When the full equilibrium is reached, Θ , μ and \mathcal{K} reduce respectively to the usual notions of temperature, chemical potential and grand potential density.

We have thus shown that isotropic out of equilibrium systems, under the assumption of a separation of two time-scales τ_M and τ_m , have an extended equation of state which is formally identical to the one of a multi-constituent single fluid: to each abstract parameter conserved in an adiabatic expansion describing the displacement from equilibrium we can associate an effective chemical species.

2.4. The five hydrodynamic regimes

The microscopic time-scale τ_m and the macroscopic time-scale τ_M introduced in subsection 2.1 also define a separation between different hydrodynamic regimes. Depending on the time-scale τ_H of the hydrodynamic process under consideration, we can identify five distinct regimes.

- *Kinetic regime*: $\tau_H \lesssim \tau_m$

In this limit there is no hope to get a closed hydrodynamic system of equations because the number of independent degrees of freedom diverges. In this case a full kinetic-theory description is required.

- *Frozen regime*: $\tau_m \ll \tau_H \approx \tau_{fr} \ll \tau_M$

In this limit the relaxation processes for the variables x_A do not have time to occur. On the other hand all the remaining microscopic degrees of freedom thermalize. This limit is characterised by the conservation of the fractions x_A and there is no entropy production.

- *Full-dissipation regime*: $\tau_H \approx \tau_M$

This is the regime of maximum dissipation, which can be seen as a chemical transfusion between the fractions x_A . The reason why, for $\tau_H \approx \tau_M$, the dissipation is maximal can be understood by recalling that, by definition, τ_M is the characteristic time-scale over which the fluid under consideration (prepared with arbitrary initial conditions) dissipates possible initial deviations from local thermodynamic equilibrium. Hence, the dissipative processes do not have time to occur (or are not particularly efficient) over time-scales $\tau_H \ll \tau_M$ (but still much longer than τ_m). In the opposite limit, when the expansions/contractions of the fluid are slower than τ_M (i.e. $\tau_H \gg \tau_M$), the reactions are fast enough to keep all the fractions close to their equilibrium value, making the entropy production inevitably small. On the contrary, when $\tau_H \approx \tau_M$, the fluid is in a sort of resonant state, where the out-of-equilibrium degrees of freedom respond on the same time-scale of the macroscopic motion, leading to efficient internal dissipation.

- *Parabolic (Navier–Stokes) regime*: $\tau_H \gg \tau_M$ and

$$\left. \frac{\partial x_A}{\partial v} \right|_{x_s, \mathbb{A}^B} \longrightarrow \infty \quad (17)$$

In this limit the relaxation time is small with respect to the hydrodynamic time-scale, however the expansion of the volume elements produces large deviations from local equilibrium. In this limit an expansion of the volume element produces instantaneously a displacement from local equilibrium and the affinity is proportional to the expansion.

- *Equilibrium regime*: $\tau_H \gg \tau_M$

In this limit the relaxation processes are so fast that local thermodynamic equilibrium is always achieved, reducing the model to a perfect fluid.

Throughout the paper these regimes will be studied in more detail one by one, but we have anticipated them here to give a schematic idea of the role of the time-scales τ_m and τ_M .

2.5. Chemical gauge of the effective currents

Consider an arbitrary coordinate transformation

$$(v, x_s, x_A) \mapsto (v, x_s, y_B), \quad (18)$$

where $A, B = 1, \dots, l-1$. This kind of transformation represents a chemical gauge-fixing of the type presented in [39] and generalised in [37]. If we write the differential of the energy per particle in the new chart we obtain

$$d\tilde{\mathcal{U}} = \Theta' dx_s + \mathcal{K}' dv - \sum_{B=1}^{l-1} \mathbb{B}^B dy_B, \quad (19)$$

where the coefficients Θ' , \mathcal{K}' and \mathbb{B}^B are related with Θ , \mathcal{K} and \mathbb{A}^A through the relations (cf (10))

$$\begin{aligned} \Theta &= \Theta' - \sum_{B=1}^{l-1} \mathbb{B}^B \frac{\partial y_B}{\partial x_s} \\ \mathcal{K} &= \mathcal{K}' - \sum_{B=1}^{l-1} \mathbb{B}^B \frac{\partial y_B}{\partial v} \\ \mathbb{A}^A &= \sum_{B=1}^{l-1} \mathbb{B}^B \frac{\partial y_B}{\partial x_A}. \end{aligned} \quad (20)$$

The partial derivatives are performed taking y_B as functions of (v, x_s, x_A) . Let us suppose that the y_B are conserved in the adiabatic expansions described in subsection 2.2. This makes them a choice of variables which is completely equivalent to the x_A . Using (9), we have

$$W_{\text{fr}(y_B)} = \frac{\partial y_B}{\partial v} = 0, \quad (21)$$

which implies

$$\mathcal{K} = \mathcal{K}'. \quad (22)$$

We have found that, even if there is not a unique way to define the x_A , this does not produce any ambiguity in the definition of \mathcal{K} . This has a deep physical origin, as we will see in subsection 3.1. Note that there is a manifestly gauge-invariant definition for \mathcal{K} :

$$\mathcal{K} = W_{\text{fr}(\tilde{\mathcal{U}})}, \quad (23)$$

to prove it one only has to apply (10) to (9).

Since the transformation (18) is a coordinate transformation, then the matrix $\partial y_B / \partial x_A$ is invertible, implying that all the $\mathbb{A}^A = 0$ if and only if all the $\mathbb{B}^B = 0$. This descends from the fact that in both the coordinate systems the requirement of vanishing affinities represents the condition of maximum entropy (constraining v and $\tilde{\mathcal{U}}$), which holds independently of the coordinate system we choose and describes the global equilibrium state.

We finally remark that our definition of the temperature is not invariant under the coordinate transformation. Since Θ might differ from Θ' only out of equilibrium, this should not be considered a serious problem, but only a particular case of the universal ambiguity of the non-equilibrium temperature, see also [40]. In fact unambiguous definitions of temperature are usually obtained involving an hypothetical equilibrium of the system with an ideal heat bath, which is in contrast with the idea of a quasi-equilibrium state, see appendix B.

2.6. Kinetics of the equilibration process

Let us finally analyse the relaxation to equilibrium of the variables x_A . The evolution far from equilibrium, for large affinities, may be in general complicated. Hence, even if up to now our discussion is correct also far from equilibrium, we focus here on the case with small \mathbb{A}^A .

Let us follow the evolution of the system on a time-scale τ_M . Assuming the walls to be blocked, the energy and the volume do not change, thus, according to the second principle of thermodynamics, we have

$$\Theta \frac{dx_s}{dt} = \sum_{A=1}^{l-1} \mathbb{A}^A \frac{dx_A}{dt} \geq 0, \quad (24)$$

where we have made use of (11). Pushing forward the analogies with chemical reactions we introduce the generalised reaction rates through the formula

$$\frac{dn_A}{dt} = r_A. \quad (25)$$

Since all the microscopic degrees of freedom apart from the x_A have a relaxation time $\tau_m \ll \tau_M$ (τ_M is the time-scale under consideration now), in each instant all the macroscopic properties of the fluid are given once the point in \mathcal{Z} is known, so that we can impose $r_A = r_A(n, \Theta, \mathbb{A}^B)$. Near equilibrium we can expand to the first order r_A for small affinities. Since $r_A(\mathbb{A}^B = 0) = 0$, we find

$$\frac{dx_A}{dt} = \frac{1}{n} \sum_{B=1}^{l-1} \Xi_{AB} \mathbb{A}^B, \quad (26)$$

where the $(l-1) \times (l-1)$ coefficients

$$\Xi_{AB} = \left. \frac{\partial r_A}{\partial \mathbb{A}^B} \right|_{\mathbb{A}^B=0} \quad (27)$$

are functions only of n and Θ . According to Onsager's principle [41, 42],

$$\Xi_{AB} = \Xi_{BA}, \quad (28)$$

therefore only $l(l-1)/2$ independent coefficients must be computed with the aid of kinetic theory. Plugging (26) in (24) we find

$$\Theta \frac{ds}{dt} = \sum_{A,B=1}^{l-1} \Xi_{AB} \mathbb{A}^A \mathbb{A}^B \geq 0, \quad (29)$$

which must be true for any small value of the \mathbb{A}^A . This implies that Ξ_{AB} has to be definite non-negative. However, accounting for the fact that there are no other constants of motion apart from N , V and E , which is a result of the so called *ergodic assumption* [43], we can replace

in the above equation the \geq with $>$, implying that Ξ_{AB} is also invertible (therefore positive definite), producing the notable constraints

$$\Xi_{AA} > 0, \quad \Xi_{AA}\Xi_{BB} > \Xi_{AB}^2 \quad \forall A \neq B. \quad (30)$$

Once the extended equation of state for quasi-equilibrium states is given through adapted equilibrium statistical mechanical calculations and the coefficients Ξ_{AB} are computed in the context of kinetic theory³, all the macroscopic properties of the system are known and it is possible to study the whole thermodynamic evolution.

In the special case in which the substance is a multi-constituent fluid and the bulk viscosity is due to the presence of chemical reactions, the x_A are reaction coordinates and \mathbb{A}^A are reaction affinities. In this case the r_A are the usual reaction rates and Ξ_{AB} are their first order expansion coefficients around equilibrium [26, 37, 44].

3. Dissipative hydrodynamics of locally isotropic fluids

We are now ready to develop the most general hydrodynamic description of a locally isotropic out-of-equilibrium fluid consistent with the principles of extended irreversible thermodynamics.

Let us assume that we can construct a current

$$n^\nu = nu^\nu, \quad (31)$$

where u is the local four-velocity and n is the rest-frame particle density, and that the continuity equation

$$\nabla_\nu n^\nu = 0 \quad (32)$$

holds. The four-velocity is normalized as $u_\nu u^\nu = -1$, so that the acceleration $a^\nu = u^\rho \nabla_\rho u^\nu$ of the fluid elements satisfies $a_\nu u^\nu = 0$.

In general it is possible to construct an energy–momentum tensor for the fluid, which in the absence of other fields appears in the right-hand side of the Einstein equations,

$$G_{\nu\rho} = 8\pi T_{\nu\rho}, \quad (33)$$

where $G_{\nu\rho}$ is the Einstein tensor, so that

$$\nabla_\rho T^\rho{}_\nu = 0. \quad (34)$$

Now, the key assumption we make is that in the comoving frame locally defined by u the matter element is isotropic. Hence, the energy–momentum tensor takes the form

$$T^{\nu\rho} = \mathcal{U}u^\nu u^\rho + \Psi h^{\nu\rho}, \quad (35)$$

where \mathcal{U} and Ψ are the energy density and the diagonal term of the stress tensor measured in the frame of u and

$$h_{\nu\rho} := g_{\nu\rho} + u_\nu u_\rho. \quad (36)$$

³ There is no way to compute the Ξ_{AB} by means of thermodynamic calculations only as thermodynamics does not study the evolution of systems, but only the properties of their macrostates seen as stationary or quasi-stationary.

The energy–momentum tensor has a perfect fluid form, so we call Ψ generalised pressure. However, at the moment no relationship between \mathcal{U} and Ψ is given because the system is not in thermal equilibrium. We can project (34) tangentially and orthogonally to u , so, using equation (35), we get

$$\dot{\mathcal{U}} + (\mathcal{U} + \Psi)\nabla_\nu u^\nu = 0 \quad a_\nu = -\frac{1}{\mathcal{U} + \Psi}h^\rho{}_\nu\nabla_\rho\Psi, \quad (37)$$

where we have introduced the notation $\dot{f} = u^\nu\partial_\nu f$ for any function f . Using (2) and (32), the first relation becomes

$$\dot{\mathcal{U}} = -\Psi\dot{v}. \quad (38)$$

This equation expresses the fact that since, by local isotropy, there is no heat flow, i.e. no energy flux in the frame of the fluid element

$$q^\nu = -T_{\rho\lambda}u^\rho h^{\lambda\nu} = 0, \quad (39)$$

all the variations of the energy per particle in this frame are the result of the work of Ψ .

3.1. Effective multifluid hydrodynamics

Let us assume that, at a kinetic level, the fluid admits the time-scale separation presented in subsection 2.2. We define the hydrodynamic time-scale as $\tau_H = L_0/c_s$, where L_0 is the length-scale in which the hydrodynamic variables change and c_s is the speed of sound. In this work we always assume that $\tau_H \gg \tau_m$, otherwise a pure hydrodynamic description would not be possible. Thus the fluid is, in each point of the space-time, in local quasi-equilibrium and the fluid elements are described by the equation of state (10). Equation (38) then becomes

$$(\mathcal{K} + \Psi)\dot{v} + \Theta\dot{x}_s - \sum_{A=1}^{l-1} \mathbb{A}^A \dot{x}_A = 0. \quad (40)$$

Now, since we are assuming that the macroscopic local properties of the fluid are completely determined once s , n and n_A are given, then an equation of state for Ψ should exist. Consider an adiabatic fast expansion with time-scale τ_{fr} . In this transformation both x_s and x_A are conserved, therefore equation (40) implies

$$\mathcal{K} = -\Psi, \quad (41)$$

in full analogy with the results of equilibrium thermodynamics. With the aid of equation (16) we obtain the Gibbs–Duhem relation

$$d\Psi = s d\Theta + n d\mu - \sum_{A=1}^{l-1} n_A d\mathbb{A}^A. \quad (42)$$

This result means that the isotropic stresses of the fluid are completely known once an equation of state for the energy density in the frame of the fluid element is given. To obtain its formula, however, we must make a Legendre transformation also with respect to the additional state variables associated with the displacement from equilibrium of the system. We remark that equation (41) holds only because we are using the chemical-like chart on \mathcal{Z} . The key ingredient, in fact, was to invoke the conservation over a time-scale τ_{fr} of x_A in equation (40), which is exactly the property the x_A were designed to satisfy.

Plugging equation (41) into (40) we find the formula

$$\Theta \dot{x}_s = \sum_{A=1}^{l-1} \mathbb{A}^A \dot{x}_A, \quad (43)$$

which describes the heat production due to the relaxation processes towards equilibrium of the variables x_A and is in agreement with (24).

Let us now study the hydrodynamic processes which occur on a time-scale τ_{fr} . At this level the transformations which the fluid elements incur are adiabatic expansions of the type described in subsection 2.2, so, again, we can impose

$$\dot{x}_s = \dot{x}_A = 0. \quad (44)$$

This means that the entropy per particle and the quantities x_A are frozen to a constant value along the worldline of the matter element. For this reason we can call this regime, in agreement with Haensel *et al* [38], *frozen regime*, see subsection 2.4. In this limit the adiabatic index describing the response of the fluid to a perturbation is

$$\Gamma_{\text{FR}} = \left. \frac{d \ln \Psi}{d \ln n} \right|_{x_s, x_A}. \quad (45)$$

We, now, define the currents

$$s^\nu = s u^\nu \quad n_A^\nu = n_A u^\nu, \quad (46)$$

which, using (32) and (44), are conserved:

$$\nabla_\nu s^\nu = 0 \quad \nabla_\nu n_A^\nu = 0. \quad (47)$$

Since \mathcal{U} is the energy density measure in the frame of the entropy, we call it *internal energy density*. It is possible to verify that the system (31), (32), (35), (14), (41), (46) and (47) describes a multifluid with l components locked to the entropy. Therefore, this system must arise directly from a convective variational principle and can be shown to be a well-posed problem, see [45]. It is also interesting to note that our result shows that in Carter's multifluid formalism the notion of *current* is a very general concept, being useful to describe not only real chemical species, but also abstract non-equilibrium variables.

3.2. Dissipative hydrodynamics

Let us assume that $\tau_{\text{H}} \gtrsim \tau_{\text{fr}}$. Then, coherently with section 2, we can also impose

$$\dot{x}_A = \frac{1}{n} \sum_{B=1}^{l-1} \Xi_{AB} \mathbb{A}^B. \quad (48)$$

Plugging this formula inside (43) we obtain

$$\Theta \dot{x}_s = \frac{1}{n} \sum_{A,B=1}^{l-1} \Xi_{AB} \mathbb{A}^A \mathbb{A}^B. \quad (49)$$

With the aid of (32), these equations can be recast in the form

$$\nabla_\nu n_A^\nu = \sum_{B=1}^{l-1} \Xi_{AB} \mathbb{A}^B \quad \Theta \nabla_\nu s^\nu = \sum_{A,B=1}^{l-1} \Xi_{AB} \mathbb{A}^A \mathbb{A}^B. \quad (50)$$

We have shown that any dissipative process in a locally isotropic fluid can be modelled as a chemical transfusion between effective currents representing a convenient choice of coordinates in the space of quasi-equilibrium states \mathcal{Z} .

Note that in the limit in which the hydrodynamic time-scale is much larger than τ_M ($\Xi_{AB} \rightarrow +\infty$), the reactions become so fast that full thermodynamic equilibrium is everywhere achieved, namely

$$\mathbb{A}^A = 0. \tag{51}$$

This implies that in this limit

$$x_A = x_A^{\text{eq}}(n, x_s) \quad \dot{x}_s = 0. \tag{52}$$

To derive the equations in this regime, one has to take the limit $\Xi_{AB} \rightarrow +\infty$ and $\mathbb{A}^A \rightarrow 0$ in the first equation of (50), keeping their product finite. Regarding the second equation, since the right-hand side is quadratic in \mathbb{A}^A and linear in Ξ_{AB} , the entropy production vanishes.

We have recovered the ideal perfect fluid and the equation of state (14) reduces to

$$d\mathcal{U} = \Theta ds + \mu dn. \tag{53}$$

In this limit, called *equilibrium regime* (see subsection 2.4), matter reacts to perturbations with the adiabatic index (cf with [38])

$$\Gamma_{\text{EQ}} = \frac{d \ln \Psi(n, x_s, x_A^{\text{eq}}(n, x_s))}{d \ln n} = \Gamma_{\text{FR}} + \frac{n}{\Psi} \sum_{A=1}^{l-1} \frac{\partial x_A^{\text{eq}}}{\partial n} \bigg|_{x_s} \frac{\partial \Psi}{\partial x_A} \bigg|_{n, x_s, x_B}. \tag{54}$$

3.3. Summary of the equations of the theory

We can finally summarize the complete set of equations of the theory, valid in all the regimes (provided that $\tau_H \gg \tau_m$). The variables are $g_{\nu\rho}$, n^ν , x_s and all the x_A . The system of differential equations is

$$\begin{aligned} G_{\nu\rho} &= 8\pi(n\tilde{\mathcal{U}}u_\nu u_\rho + \Psi h_{\nu\rho}) \\ \nabla_\nu n^\nu &= 0 \\ u^\nu \nabla_\nu x_s &= \frac{1}{n\Theta} \sum_{A,B=1}^{l-1} \Xi_{AB} \mathbb{A}^A \mathbb{A}^B \\ u^\nu \nabla_\nu x_A &= \frac{1}{n} \sum_{B=1}^{l-1} \Xi_{AB} \mathbb{A}^B. \end{aligned} \tag{55}$$

In addition, it is necessary to have an equation of state $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}(n^{-1}, x_s, x_A)$ and an expression for the transport coefficients $\Xi_{AB} = \Xi_{AB}(n, x_s)$.

In the case in which one wants to work with fixed background spacetime, then $g_{\nu\rho}$ is no more a variable and Einstein's equations can be replaced by the Euler equation

$$(n\tilde{\mathcal{U}} + \Psi)u^\rho \nabla_\rho u_\nu = -h^\rho_\nu \nabla_\rho \Psi. \tag{56}$$

4. Emergence of bulk viscosity

In this section we prove that the hydrodynamic model developed in the previous section describes, in the first order in \mathbb{A}^A , a locally isotropic viscous fluid, i.e. a perfect fluid with a bulk viscosity term. We show, however, that the model has a natural hyperbolic form, being governed by a telegraph-type equation, whose relaxation time-scale is given by τ_M . We will examine the parabolic limit for arbitrary l ⁴, proving that the relativistic Navier–Stokes formulation is recovered. Finally we will summarize how the different regimes can be obtained varying the time-scales of the involved processes.

From now on we will use the Einstein summation convention for repeated indices A, B, C, D .

4.1. Expansion for small deviations from equilibrium: thermodynamic potentials

It is convenient to consider the quantities per particle and use as free state variables x_s, v and \mathbb{A}^A . The affinities are preferable in this context to x_A , because our aim is to make an expansion for small displacements from equilibrium. In this section all the quantities are assumed to be functions of these variables and the partial derivatives will be performed accordingly.

Recalling equation (10) and the identification (41), we can introduce the new thermodynamic potential

$$\tilde{\mathcal{G}} := \tilde{\mathcal{U}} + \mathbb{A}^A x_A, \tag{57}$$

whose differential is

$$d\tilde{\mathcal{G}} = \Theta dx_s - \Psi dv + x_A d\mathbb{A}^A. \tag{58}$$

The perturbative approach is built by expanding $\tilde{\mathcal{G}}$ to second order:

$$\tilde{\mathcal{G}}(x_s, v, \mathbb{A}^A) = \tilde{\mathcal{U}}_{\text{eq}} + x_A^{\text{eq}} \mathbb{A}^A + \frac{1}{2} \tilde{\mathcal{G}}_{AB}^{\text{eq}} \mathbb{A}^A \mathbb{A}^B, \tag{59}$$

where the coefficients $\tilde{\mathcal{U}}_{\text{eq}}, x_A^{\text{eq}}$ and $\tilde{\mathcal{G}}_{AB}^{\text{eq}}$ are functions only of x_s and v . The zeroth order term in the expansion is the energy per particle evaluated in equilibrium,

$$\tilde{\mathcal{U}}_{\text{eq}} = \tilde{\mathcal{U}}|_{\mathbb{A}^A=0} = \tilde{\mathcal{G}}|_{\mathbb{A}^A=0}, \tag{60}$$

and the first order expansion coefficients are the equilibrium fractions

$$x_A^{\text{eq}} = x_A|_{\mathbb{A}^B=0} = \left. \frac{\partial \tilde{\mathcal{G}}}{\partial \mathbb{A}^A} \right|_{\mathbb{A}^B=0}. \tag{61}$$

The $(l-1) \times (l-1)$ matrix $\tilde{\mathcal{G}}_{AB}^{\text{eq}}$ in (59) is defined as

$$\tilde{\mathcal{G}}_{AB}^{\text{eq}} = \left. \frac{\partial^2 \tilde{\mathcal{G}}}{\partial \mathbb{A}^A \partial \mathbb{A}^B} \right|_{\mathbb{A}^C=0}. \tag{62}$$

Defining

$$\Theta_{\text{eq}} := \frac{\partial \tilde{\mathcal{U}}_{\text{eq}}}{\partial x_s} \quad P := - \frac{\partial \tilde{\mathcal{U}}_{\text{eq}}}{\partial v} \tag{63}$$

⁴We recall that l counts the number of degrees of freedom (other than n) that have to be included in the quasi-equilibrium equation of state together with the number density.

and considering equation (58), we obtain

$$\begin{aligned} \Theta &= \Theta_{\text{eq}} + \mathbb{A}^A \frac{\partial x_A^{\text{eq}}}{\partial x_s} + \frac{1}{2} \mathbb{A}^A \mathbb{A}^B \frac{\partial \tilde{\mathcal{G}}_{AB}^{\text{eq}}}{\partial x_s} \\ \Psi &= P - \mathbb{A}^A \frac{\partial x_A^{\text{eq}}}{\partial v} - \frac{1}{2} \mathbb{A}^A \mathbb{A}^B \frac{\partial \tilde{\mathcal{G}}_{AB}^{\text{eq}}}{\partial v} \\ x_A &= x_A^{\text{eq}} + \tilde{\mathcal{G}}_{AB}^{\text{eq}} \mathbb{A}^B. \end{aligned} \tag{64}$$

The quantity P is the *equilibrium pressure*, the quantity

$$\Pi = -\mathbb{A}^A \frac{\partial x_A^{\text{eq}}}{\partial v} \tag{65}$$

is called *first-order viscous stress* and the quantity

$$\Pi'' = -\frac{1}{2} \mathbb{A}^A \mathbb{A}^B \frac{\partial \tilde{\mathcal{G}}_{AB}^{\text{eq}}}{\partial v} \tag{66}$$

is called *second-order viscous stress*. Therefore the second equation of (64) can be rewritten in the form

$$\Psi = P + \Pi + \Pi'', \tag{67}$$

in which P can be considered the thermodynamic pressure (in the sense that it can be computed directly from equilibrium thermodynamics), while Π and Π'' constitute the first two contributions to the bulk viscosity.

Now we can reverse the Legendre transformation (57) and get

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_{\text{eq}} - \frac{1}{2} \tilde{\mathcal{G}}_{AB}^{\text{eq}} \mathbb{A}^A \mathbb{A}^B, \tag{68}$$

so the first order correction in \mathbb{A}^A to the internal energy density is zero. This formula can also be used to prove that

$$\tilde{\mathcal{G}}_{AB}^{\text{eq}} = \Theta_{\text{eq}} \frac{\partial^2 x_s}{\partial \mathbb{A}^A \partial \mathbb{A}^B} \Big|_{v, \tilde{\mathcal{U}}, \mathbb{A}^C=0}. \tag{69}$$

Therefore the matrix $\tilde{\mathcal{G}}_{AB}^{\text{eq}}$ must be negative definite, implying that equilibrium is the minimum of the energy per particle with fixed v and x_s , in accordance with [46].

Up to now we have considered corrections to the second order, to keep track of the corrections induced on all the thermodynamic potentials. The first step to recover the usual formulation for the bulk viscosity is to consider the formula (35) for the energy–momentum tensor of the system and truncate the expansion to the first order in \mathbb{A}^A . This leads us to

$$T^{\nu\rho} = \mathcal{U}_{\text{eq}} u^\nu u^\rho + (P + \Pi) h^{\nu\rho}, \tag{70}$$

where $\mathcal{U}_{\text{eq}} = n \tilde{\mathcal{U}}_{\text{eq}}$. We obtained the perfect fluid in local thermodynamic equilibrium, described by the equation of state $\mathcal{U}_{\text{eq}}(s, n)$, plus a bulk-viscosity correction Π to the isotropic stresses.

4.2. Expansion for small deviations from equilibrium: dissipation

We have verified that for small perturbations from equilibrium, the energy–momentum tensor takes the usual form for a bulk-viscous fluid. To complete the perturbative expansion of the theory we need to study (48) in the limit of small affinities. Let us consider x_A as a function of x_s, v and \mathbb{A}^A , in accordance with what we did in subsection 4.1. Then, using the chain rule, we have that equation (48) becomes

$$\frac{\partial x_A}{\partial x_s} \dot{x}_s + \frac{\partial x_A}{\partial v} \dot{v} + \frac{\partial x_A}{\partial \mathbb{A}^B} \dot{\mathbb{A}}^B = \frac{1}{n} \Xi_{AB} \mathbb{A}^B. \tag{71}$$

Now, equation (32) implies that

$$\dot{v} = v \nabla_\nu u^\nu, \tag{72}$$

so, contracting (71) with the symmetric $(l - 1) \times (l - 1)$ matrix Ξ^{AB} , defined as the inverse of Ξ_{AB} ,

$$\Xi^{AC} \Xi_{CB} = \delta^A_B, \tag{73}$$

we find

$$-n \Xi^{AB} \frac{\partial x_B}{\partial \mathbb{A}^C} \dot{\mathbb{A}}^C + \mathbb{A}^A - n \Xi^{AB} \frac{\partial x_B}{\partial x_s} \dot{x}_s = \Xi^{AB} \frac{\partial x_B}{\partial v} \nabla_\nu u^\nu. \tag{74}$$

Now we define the $(l - 1) \times (l - 1)$ matrix

$$\tau^A_C = -n \Xi^{AB} \frac{\partial x_B}{\partial \mathbb{A}^C}, \tag{75}$$

called *relaxation-time matrix*, the $l - 1$ vector

$$k^A = \Xi^{AB} \frac{\partial x_B}{\partial v} \tag{76}$$

and the second order $l - 1$ vector, see (49),

$$\mathcal{Q}^A = -n \Xi^{AB} \frac{\partial x_B}{\partial x_s} \dot{x}_s = -\Xi^{AB} \Xi_{CD} \frac{\partial x_B}{\partial x_s} \frac{\mathbb{A}^C \mathbb{A}^D}{\Theta}. \tag{77}$$

Therefore equation (74) can be rewritten in the form

$$\tau^A_B \dot{\mathbb{A}}^B + \mathbb{A}^A + \mathcal{Q}^A = k^A \nabla_\nu u^\nu. \tag{78}$$

The final step consists of keeping only the lowest order in the affinities, which means that the transport coefficients τ^A_B and k^A can be evaluated in $\mathbb{A}^A = 0$ (thus we can put a superscript eq to the quantities appearing in the right-hand side of their definitions) and \mathcal{Q}^A is approximated to zero, leaving the telegraph-type equation

$$\tau^A_B \dot{\mathbb{A}}^B + \mathbb{A}^A = k^A \nabla_\nu u^\nu. \tag{79}$$

The first term of left-hand side encodes the relaxation time-scale, clearly the matrix τ^A_B is of first order in τ_M . The right-hand side describes the fact that if the volume element expands, then the fluid is driven out of local thermodynamic equilibrium if the relaxation processes are not sufficiently fast. In particular, k^A quantifies the response of the affinity \mathbb{A}^A to the expansion.

In relativity, the need to have telegraph-type equations describing the evolution of the internal degrees of freedom of the matter elements is well established [17, 31, 45, 47]. However, the relaxation term, usually inserted ad-hoc to make the theories hyperbolic⁵, has been regarded by some authors more as an artefact imposed to fulfil a mathematical necessity, rather than a physically justified contribution [16]. Thus, equation (79) represents a justification of the existence of this term, arising directly from arguments of non-equilibrium thermodynamics. Our approach, however, differs from previous ones (see [21] for a summary), because the telegraph-type equation is derived for the more fundamental quantities \mathbb{A}^A and not directly for Π .

4.3. The parabolic limit

An important test is to verify if we can recover the usual Navier–Stokes prediction for the bulk viscosity in an appropriate limit.

In equation (79) the term $\tau^A{}_B \dot{\mathbb{A}}^B$ describes a delay in the response of matter to an expansion. To recover Navier–Stokes we have to assume that this term is negligible. The conditions we need to achieve our goal are the following:

$$\tau_M \longrightarrow 0 \quad \frac{\partial x_A}{\partial v} \longrightarrow \infty. \tag{80}$$

In fact we need to impose that τ_M is smaller than the hydrodynamic time-scale, sufficiently short to assume that there is no long-term memory of the past, but not so small to recover the frozen regime. In particular, we need that the quantities k^A , given in (76), remain finite. Since

$$\Xi^{AB} \longrightarrow 0, \tag{81}$$

we need to compensate imposing the second condition of (80).

In this limit we can make the approximation

$$\mathbb{A}^A = k^A \nabla_\nu u^\nu. \tag{82}$$

Then, using (65), we obtain

$$\Pi = -\zeta \nabla_\nu u^\nu, \tag{83}$$

with

$$\zeta = \Xi^{AB} \frac{\partial x_A^{\text{eq}}}{\partial v} \frac{\partial x_B^{\text{eq}}}{\partial v}. \tag{84}$$

Since Ξ_{AB} is positive definite, so is Ξ^{AB} , and therefore

$$\zeta \geq 0. \tag{85}$$

⁵ See, e.g. [48], for an introduction to hyperbolicity and parabolicity, with reference to the telegraph equation. Intuitively, the difference is in the fact that, in a parabolic system, the domain of influence of the initial data imposed on a point is bounded (locally) by a 3D hyperplane (giving rise to acausal dynamics) while, in a hyperbolic system, it is bounded by a 3D cone and can therefore be causal if this cone is contained inside the light-cone. The telegraph equation is the prototype of a dissipative hyperbolic equation.

We conclude analysing the entropy production in the parabolic limit. If we plug (82) in (49) we find

$$\Theta \dot{x}_s = \frac{1}{n} \Xi_{AB} k^A k^B (\nabla_\nu u^\nu)^2. \quad (86)$$

This can be recast into the best known formula

$$\Theta \nabla_\nu s^\nu = -\Pi \nabla_\nu u^\nu = \zeta \left(\frac{\dot{v}}{v} \right)^2 \geq 0. \quad (87)$$

Therefore we have recovered all the equations of the relativistic Navier–Stokes model, which is a parabolic system, see [16]. We are using the general term relativistic Navier–Stokes model to denote the *first order theories* of Eckart [49], Landau and Lifshitz [50] and more in general those considered by Hiscock and Lindblom [17]. They all converge to the unique model we presented above in the case of bulk-viscous substances, in the absence of heat flux and shear viscosity.

We remark that in the hyperbolic regime, as l grows, the model becomes increasingly complicated because all the out-of-equilibrium variables have their own equilibration time and can influence the evolution of each other. On the other hand, in the parabolic limit all the contributions add up in a unique factor ζ in which it is not possible to distinguish the individual microphysical processes.

We conclude this section by expanding on the physical meaning of the limits in (80). The limit $\tau_M \rightarrow 0$ is a requirement of slow evolution of the system. This condition is also commonly used to derive the Navier–Stokes equations directly from kinetic theory [51]. Since the instability of relativistic Navier–Stokes arises from the existence of fast-growing unphysical modes [17, 18, 48], the condition of slow evolution guarantees that the telegraph-type equation (79) is equivalent to (82) only for physical solutions of the latter. In contrast, the theory behaves differently along the gapped modes of the system, where first-order theories become unphysical [52] and the relaxation term $\tau^A_B \dot{\Delta}^B$ cannot be neglected.

The second limit of (80) is a formal way of stating that, for bulk viscosity to survive the slow limit, some $x_A^{\text{eq}}(v, x_s)$ need to have a strong dependence on v : they should vary considerably as the volume elements expand, forcing the fluid to dissipate energy in the attempt to re-equilibrate the fractions x_A to the value x_A^{eq} . Only in this case the entropy production (87) is not negligible, even if it is a second-order infinitesimal (due to the slow limit applied to the expansion rate \dot{v}/v).

5. Heat production in small oscillations

In this section we study the damping of oscillations of homogeneous systems produced by a bulk viscosity term. We show that, as our system is hyperbolic and described by telegraph-type equation (79), it naturally produces a dependence on the frequency of the oscillations related to the delayed response of matter on a time-scale τ_M , unlike the standard parabolic Navier–Stokes approach. Our theory thus generally includes this effect, which was studied by Sawyer [53] for the specific case of bulk viscosity due to reactions in neutron stars.

5.1. Setting the stage

To study a weak damping effect on small oscillations around a uniform equilibrium configuration we solve the linearised dynamics for perturbations in the non-dissipative limit (i.e. by replacing the bulk-viscous fluid with the perfect fluid that we would recover if the

system was in the equilibrium regime: $\mathbb{A}^A = 0$). Then, we compute separately the evolution of the affinities \mathbb{A}^A (at the first-order) and the associated heat production (which is treated as a second-order correction), assuming that they have a negligible effect (small back-reaction limit) on the dynamics of the oscillations for the regimes and associated time-scales that are being considered [53, 54].

Hence, the first step consists of studying the oscillations of a perfect fluid in local thermodynamic equilibrium around a homogeneous static solution (the independent variables are 5, so one equation is redundant),

$$\begin{aligned}\nabla_\nu n^\nu &= 0 \\ \nabla_\nu s^\nu &= 0 \\ u^\rho \nabla_\rho u_\nu &= -\frac{1}{\mathcal{U}_{\text{eq}} + P} h^\rho{}_\nu \nabla_\rho P.\end{aligned}\tag{88}$$

In addition we have the equation of state $\mathcal{U}_{\text{eq}}(n, s)$ and we work in the flat spacetime limit. We use the global inertial chart (t, z^1, z^2, z^3) , we assume invariance under translations in the directions 2 and 3 and we impose $u^2 = u^3 = 0$. Any physical quantity X is assumed to have a spacetime dependence of the form

$$X = X_0 + \delta X e^{i(kz^1 - \omega t)},\tag{89}$$

where X_0 is the unperturbed value and δX is a complex amplitude of the perturbation (both are uniform and constant) all the fluctuations are encoded in the exponential.

The system describes the propagation of sound waves in the medium:

$$\delta v = -v \frac{k}{\omega} \delta u^1 \quad \delta x_s = 0 \quad \frac{\omega}{k} = c_s,\tag{90}$$

where

$$c_s = \sqrt{\left. \frac{-v}{\mathcal{U}_{\text{eq}} + P} \frac{\partial P}{\partial v} \right|_{x_s}} = \sqrt{\left. \frac{\partial P}{\partial \mathcal{U}_{\text{eq}}} \right|_{x_s}}\tag{91}$$

is the speed of sound. The perturbations to the affinities will be computed explicitly in the next subsections, assuming that (90) is approximately valid also in the bulk-viscous case and invoking the telegraph-type equation (79). For the time being, we keep the discussion general, leaving the values of $\delta \mathbb{A}^A$ (and consequently of δx_A) undetermined.

Now we compute the energy dissipated in a large volume V and in a time τ_d which is assumed to be large with respect to ω^{-1} (in order to contain a large number of oscillations), but sufficiently small to neglect the back-reaction of dissipation on the oscillations. The heat produced is

$$\Delta Q = \Theta_0 \Delta S = \int_0^{\tau_d} \int_V \Xi_{AB} \mathbb{A}^A \mathbb{A}^B d_3z dt.\tag{92}$$

Since

$$\mathbb{A}_0^A = 0 \quad \text{and} \quad \mathbb{A}^A = \text{Re} \left[\delta \mathbb{A}^A e^{i(kz^1 - \omega t)} \right],\tag{93}$$

the heat ΔQ is a second order effect in the fluctuations and we can replace Ξ_{AB} with Ξ_{AB}^0 in (92). In the integration the oscillating terms (in space and time) give a negligible contribution,

while the uniform terms factorise out. Thus, the average heat production per unit volume and time is

$$\frac{dq_{av}}{dt} := \frac{\Delta Q}{V\tau_d} = \frac{1}{2} \Xi_{AB}^0 \delta \mathbb{A}^A (\delta \mathbb{A}^B)^*. \quad (94)$$

Note that we should extract the real part in the right-hand side, but it is already real considering that Ξ_{AB}^0 is a real symmetric matrix.

5.2. Heat production in the parabolic limit

In the parabolic limit we need to perturb the equation (82), taking n , s and u to be given by (90), i.e. solutions of the system in the non-dissipative limit. Therefore, at first order

$$\delta \mathbb{A}^A = -i\omega \frac{\delta v}{v_0} k_0^A. \quad (95)$$

Plugging this formula into (94) we find

$$\frac{dq_{av}}{dt} = \frac{\omega^2 \delta v^2}{2v_0^2} \Xi_{AB}^0 k_0^A k_0^B. \quad (96)$$

With the aid of (76) and (84) we rewrite the above expression in the form

$$\frac{dq_{av}}{dt} = \frac{\omega^2 \delta v^2}{2v_0^2} \zeta. \quad (97)$$

5.3. Heat production in a generic regime

In an arbitrary regime (with $\tau_H \gg \tau_m$) we need to perturb the telegraph-type equation (79), obtaining

$$(\delta^A_B - i\omega \tau^A_B) \delta \mathbb{A}^B = -i\omega \frac{\delta v}{v_0} k_0^A, \quad (98)$$

where we have omitted the subscript 0 in τ^A_B . We define the matrix \mathcal{M}^A_B to be the inverse of $\delta^A_B - i\omega \tau^A_B$, which is proven to always exist in appendix A2, so we have

$$\delta \mathbb{A}^A = -i\omega \frac{\delta v}{v_0} \mathcal{M}^A_B k_0^B. \quad (99)$$

Plugging this into (94) we find

$$\frac{dq_{av}}{dt} = \frac{\omega^2 \delta v^2}{2v_0^2} \Xi_{AB}^0 \mathcal{M}^A_C (\mathcal{M}^B_D)^* k_0^C k_0^D. \quad (100)$$

By comparison with the parabolic limit it is natural to define an effective frequency-dependent bulk viscosity coefficient

$$\zeta_{\text{eff}} = \Xi_{AB}^0 \mathcal{M}^A_C (\mathcal{M}^B_D)^* k_0^C k_0^D. \quad (101)$$

Remembering that τ^A_B is of the order of τ_M , we obtain that in the limit $\omega \tau_M \rightarrow 0$ (which corresponds to $\tau_M \ll \tau_H$) $\mathcal{M}^A_B \rightarrow \delta^A_B$ and therefore $\zeta_{\text{eff}} \rightarrow \zeta$ and we recover the results given in subsection 5.2. On the other hand, in the limit $\omega \tau_M \rightarrow +\infty$ (which corresponds to

$\tau_M \gg \tau_H$) ζ_{eff} scales as ω^{-2} . The physical explanation for this is that we are in the frozen limit when the oscillations are that fast. In fact, perturbing equation (48),

$$\delta x_A = \frac{i \Xi_{AB}^0 \delta \mathbb{A}^B}{n_0 \omega} = \delta v \Xi_{AB}^0 \mathcal{M}^B{}_C k_0^C, \quad (102)$$

which goes to zero as $\omega \tau_M \rightarrow +\infty$. Since the production of entropy depends only on the average displacement of the fractions from equilibrium, which in this limit is determined only by the amplitude of the oscillation and not by its frequency, we have that dq_{av}/dt must approach a constant value for large ω , giving the condition $\omega^2 \zeta_{\text{eff}} \approx \text{const}$.

Equation (101) is the generalization to arbitrary l (and arbitrary microscopic origin) of equation (10) of Sawyer [53]. Note that the dependence on the frequency of ζ_{eff} does not disappear in the Newtonian limit, but constitutes the general thermodynamic explanation of the dependence on the frequency of the bulk viscosity noted by Landau and Lifshitz [50] and Meador *et al* [55].

6. Bulk viscosity in neutron stars

Among the possible applications of our theory, a relevant one would be the study of the chemically induced bulk viscosity in neutron stars. Despite the fact that our formalism only provides the general form of the hydrodynamic equations, it can be adapted to include any kind of nuclear reaction which contributes to the bulk viscosity. In general, one should select the relevant chemical species (the number of chemical species will coincide with the number l), provide an equation of state (valid also out of chemical equilibrium) and give a formula for the reaction matrix Ξ_{AB} (to do this one needs to expand the nuclear-reaction rates in the affinities). Then, our machinery can be employed to compute the coefficients $\tau^A{}_B$ and k^A of the hyperbolic near-equilibrium model (see subsection 4.2) or, directly, the bulk viscosity ζ . To give a practical example of how this procedure works, we will focus, here, on a minimal model in which dissipation is the product of beta-reactions.

We take a two-component model, with number densities n_p of protons and n_n of neutrons. We require the fluid to be electrically neutral, so the density of electrons is not an independent degree of freedom. The equation of state is $\mathcal{U} = \mathcal{U}(s, n_p, n_n)$ whose differential is

$$d\mathcal{U} = \Theta ds + \mu_n dn_n + \mu_p dn_p. \quad (103)$$

Neglecting superfluidity and heat flux, all the components are comoving with the entropy:

$$s^\nu = s u^\nu \quad n_p^\nu = n_p u^\nu \quad n_n^\nu = n_n u^\nu. \quad (104)$$

As a result of β reactions a particle of type p can be converted into a particle of type n and vice-versa, but the current

$$n^\nu = n_p^\nu + n_n^\nu \quad (105)$$

is conserved. The differential (103) can be rewritten in the form

$$d\mathcal{U} = \Theta ds + \mu_n dn - \mathbb{A} dn_p, \quad (106)$$

where

$$\mathbb{A} = \mu_n - \mu_p \quad (107)$$

is the affinity of the reaction. Calling the fraction of p particles $x_p = n_p/n$, the differential of the energy per-particle is

$$d\tilde{\mathcal{U}} = \Theta dx_s - \Psi dv - \mathbb{A} dx_p, \quad (108)$$

which is presented in the form (10). We now rewrite the differential $d\tilde{\mathcal{U}}$ performing a different ‘chemical choice’: the differential (103) could be equivalently given in the form

$$d\mathcal{U} = \Theta ds + \mu_p dn - \mathbb{B} dn_n, \quad (109)$$

with $\mathbb{B} = \mu_p - \mu_n = -\mathbb{A}$. Then, introducing the fraction of free neutrons $x_n = n_n/n = 1 - x_p$, we would arrive at

$$d\tilde{\mathcal{U}} = \Theta dx_s - \Psi dv - \mathbb{B} dx_n. \quad (110)$$

This gauge fixing preserves Θ and Ψ , but not the chemical potential associated to the baryon current, which in the first case is μ_n , in the second case is μ_p .

The equation of evolution of the dynamical fractions is

$$\dot{x}_p = \frac{1}{n} \Xi \mathbb{A}. \quad (111)$$

In the limit of small affinities the telegraph-type equation reads

$$\tau_M \dot{\mathbb{A}} + \mathbb{A} = k \nabla_\nu u^\nu, \quad (112)$$

where, since the relaxation time matrix has only one element, we identified it with τ_M itself:

$$\tau_M = - \frac{n}{\Xi} \frac{\partial x_p}{\partial \mathbb{A}} \Big|_{\mathbb{A}=0, v, x_s}. \quad (113)$$

The quantity k is

$$k = \frac{1}{\Xi} \frac{\partial x_p^{\text{eq}}}{\partial v} \Big|_{x_s}. \quad (114)$$

We can finally compute the formula for the effective bulk viscosity (101) by considering that in the case $l = 2$ the matrix \mathcal{M}^A_B reduces to a single coefficient

$$\mathcal{M} = \frac{1}{1 - i\omega\tau_M}. \quad (115)$$

We immediately find that

$$\zeta_{\text{eff}} = \frac{\zeta}{1 + \omega^2\tau_M^2}, \quad (116)$$

which is in accordance with Sawyer [53].

7. Recovering Israel–Stewart

In this section we show how, starting from the general theory for $l = 2$ (and only in this case)⁶, one can recover the description for bulk viscosity of Israel and Stewart [20].

7.1. Expanding the entropy

In section 4 we have expanded the thermodynamic potential $\tilde{\mathcal{G}}$ for small affinities obtaining (in the first order limit) a hyperbolic version of relativistic Navier–Stokes with bulk viscosity. However, since the approach presented in sections 2 and 3 is completely general, we expect it to reproduce also the Israel–Stewart description of bulk viscosity, taking the second order of the theory, and imposing $l = 2$.

Let us introduce, for notational convenience, the quantities

$$\beta := \frac{1}{\Theta} \quad \psi := \frac{\Psi}{\Theta} \quad a := \frac{\Lambda}{\Theta}. \tag{117}$$

Equation (11) can be rewritten in the form (for $l = 2$)

$$dx_s = \beta d\tilde{\mathcal{U}} + \psi dv + a dx. \tag{118}$$

We introduce the new state variable

$$y_s = x_s - ax, \tag{119}$$

whose differential is

$$dy_s = \beta d\tilde{\mathcal{U}} + \psi dv - x da. \tag{120}$$

Analogously to what we did in section 4, we expand y_s to the second order in a :

$$y_s = x_s^{\text{eq}} - x^{\text{eq}}a + \frac{1}{2}y_s''a^2, \tag{121}$$

where x_s^{eq} , x^{eq} and y_s'' are only functions of $\tilde{\mathcal{U}}$ and v . In the above formula we have used the fact that

$$x_s^{\text{eq}} = x_s|_{a=0} = y_s|_{a=0} \tag{122}$$

and

$$x^{\text{eq}} = x|_{a=0} = - \left. \frac{\partial y_s}{\partial a} \right|_{a=0}. \tag{123}$$

Now we introduce the quantities

$$\beta^{\text{eq}} := \frac{\partial x_s^{\text{eq}}}{\partial \tilde{\mathcal{U}}} \quad \psi^{\text{eq}} := \frac{\partial x_s^{\text{eq}}}{\partial v}. \tag{124}$$

Note that in section 4 the equilibrium quantities were defined as state variables of an hypothetical fluid in equilibrium with the same density and entropy per-particle, while now the

⁶For this reason we may say that, if for $l = 0$ we have the barotropic perfect fluid, for $l = 1$ the perfect fluid, then for $l = 2$ we have the Israel–Stewart bulk-viscous fluid (we remind that l gives the number of variables appearing in the equation of state apart from the baryon density).

hypothetical equilibrium configuration has the same density and energy per-particle. Therefore, at the second order $\beta^{\text{eq}} \neq \Theta_{\text{eq}}^{-1}$ and $\psi^{\text{eq}} \neq P/\Theta_{\text{eq}}$, where Θ_{eq} and P have been introduced in equation (63). Equation (68), however, can be easily used to prove that they coincide at the first order. Comparing (121) with (120) we find that

$$\begin{aligned} \beta &= \beta^{\text{eq}} - a \frac{\partial x^{\text{eq}}}{\partial \mathcal{U}} + \frac{a^2}{2} \frac{\partial y_s''}{\partial \mathcal{U}} \\ \psi &= \psi^{\text{eq}} - a \frac{\partial x^{\text{eq}}}{\partial v} + \frac{a^2}{2} \frac{\partial y_s''}{\partial v} \\ x &= x^{\text{eq}} - y_s'' a. \end{aligned} \tag{125}$$

Now we can reverse the Legendre transformation (119) and get

$$x_s = x_s^{\text{eq}} - \frac{1}{2} y_s'' a^2. \tag{126}$$

Note that, since $a = 0$ defines the maximum of the entropy, we immediately have

$$y_s'' > 0. \tag{127}$$

Defining $s_{\text{eq}} := n x_s^{\text{eq}}$ and using the definition of a (117) we find

$$s = s_{\text{eq}} - \frac{n y_s''}{2 \Theta^2} \mathbb{A}^2. \tag{128}$$

Now, to compare equation (128) with the expansion of Israel and Stewart [45], we need to use the quantity Π as a free variable in the equation of state of the entropy. In our approach, Π is not a fundamental thermodynamic quantity, but it is the first order correction to Ψ when the matter element goes out of equilibrium. Now, there is the complication that the reference equilibrium state which is considered in section 4 is different from the one assumed in this section. Luckily, this does not produce any confusion in the definition of Π , because this distinction emerges only at the second order. Therefore, we can still employ equation (65), imposing $l = 2$, and find

$$\Pi = -\mathbb{A} \left. \frac{\partial x^{\text{eq}}}{\partial v} \right|_{x_s}, \tag{129}$$

which, plugged into (128), gives

$$s = s_{\text{eq}} - \frac{\chi \Pi^2}{2 \Theta}, \tag{130}$$

with

$$\chi = \frac{n y_s''}{\Theta} \left(\left. \frac{\partial x^{\text{eq}}}{\partial v} \right|_{x_s} \right)^{-2} \geq 0. \tag{131}$$

This is the second order expansion of the entropy in terms of Π proposed in the Israel–Stewart formulation, see [45]. We remark that they denote our χ with the symbol β_0 , but we have changed notation to avoid confusion with the inverse temperature.

The key step in the construction of a bridge between the formalisms is to find an algebraic relationship between the parameter χ introduced in Israel–Stewart theory, and the coefficients

presented in section 4. This can be done deriving the third equation of (125) with respect to \mathbb{A} at constant v and x_s , obtaining

$$\left. \frac{\partial x}{\partial \mathbb{A}} \right|_{v, x_s} = -\frac{y_s''}{\Theta}. \tag{132}$$

Plugging this result into (131), with the aid of (75) and (84), we obtain the formula

$$\tau_M = \chi \zeta. \tag{133}$$

7.2. Telegraph-type equation for the viscous stress

Equation (79) for the case $l = 2$ reads

$$\tau_M \dot{\mathbb{A}} + \mathbb{A} = k \nabla_\nu u^\nu. \tag{134}$$

Using the definition for the transport coefficients k and ζ , equations (76) and (84), and the formula (129), we get

$$\Pi = -\zeta \nabla_\nu u^\nu + \tau_M \dot{\mathbb{A}} \left. \frac{\partial x^{\text{eq}}}{\partial v} \right|_{x_s}, \tag{135}$$

which can be rewritten in the form

$$\Pi = -\zeta \nabla_\nu u^\nu - \tau_M \dot{\Pi} - \tau_M \mathbb{A} u^\nu \partial_\nu \left(\left. \frac{\partial x^{\text{eq}}}{\partial v} \right|_{x_s} \right). \tag{136}$$

Neglecting higher order terms, we use (133) to rewrite the above equation in the form

$$\Pi = -\zeta (\nabla_\nu u^\nu + \chi \dot{\Pi}) - \mathbb{A} \zeta \nabla_\nu u^\nu \left. \frac{\partial^2 x^{\text{eq}}}{\partial v^2} \right|_{x_s} \chi v. \tag{137}$$

The last term in the right-hand side is proportional to the product between \mathbb{A} and $\zeta \nabla_\nu u^\nu$, therefore it is a higher order with respect to the other terms and we can neglect it. In the end we obtain the equation

$$\Pi = -\zeta (\nabla_\nu u^\nu + \chi \dot{\Pi}), \tag{138}$$

which completes the bridge between our formulation and the one of Israel and Stewart [20]. We remark that it has been shown in [32] that this model for bulk viscosity is causal.

7.3. Israel–Stewart modelling of neutron star bulk viscosity

To complete our discussion let us show how the coefficients of the Israel–Stewart expansion above can be computed from a multifluid approach for the case of neutron star matter presented in section 6.

The viscous stress Π , which in [20] is treated as a fundamental variable, can be written in terms of quantities appearing in the two-fluid model as

$$\Pi = (\mu_p - \mu_n) \left. \frac{\partial x_p^{\text{eq}}}{\partial v} \right|_{x_s}, \tag{139}$$

see equation (65). The transport coefficient ζ is given in general by the formula (84), which in our case reduces to

$$\zeta = \frac{1}{\Xi} \left(\frac{\partial x_p^{\text{eq}}}{\partial v} \Big|_{x_s} \right)^2, \quad (140)$$

where Ξ has been introduced in (111).⁷ In the literature it is possible to find this formula written in terms of a different set of thermodynamic quantities. Let us define the neutron excess as $\alpha = x_n - x_p$ and consider the equation

$$\mathbb{A} = \mathbb{A}(n, \alpha, x_s). \quad (141)$$

If we derive it along the curve

$$\alpha = \alpha^{\text{eq}} \quad x_s = \text{const}, \quad (142)$$

we find the thermodynamic relation

$$\frac{\partial x_p^{\text{eq}}}{\partial v} \Big|_{x_s} = - \frac{n^2 \partial_n \mathbb{A} |_{\alpha, x_s}}{2 \partial_\alpha \mathbb{A} |_{n, x_s}}, \quad (143)$$

where from now on, in this subsection, everything is computed in equilibrium, i.e. for $\alpha = \alpha^{\text{eq}}$. Plugging (143) into (140), we obtain

$$\zeta = \frac{n^4 (\partial_n \mathbb{A} |_{\alpha, x_s})^2}{4 \Xi (\partial_\alpha \mathbb{A} |_{n, x_s})^2}. \quad (144)$$

The times-scale τ_M is given in (113) and can be rewritten in the form

$$\tau_M = \frac{n}{2 \Xi \partial_\alpha \mathbb{A} |_{n, x_s}}. \quad (145)$$

If we plug (144) and (145) into (116) we get the formula for ζ_{eff} that you can find in [53, 57].

Finally, we can use equation (133) to compute the coefficient χ and get

$$\chi = \frac{2 \partial_\alpha \mathbb{A} |_{n, x_s}}{n^3 (\partial_n \mathbb{A} |_{\alpha, x_s})^2}. \quad (146)$$

8. Connection to microphysics: ideal gases

We are now ready to connect the hydrodynamic description developed in sections 2 and 3 directly with a kinetic theory of ideal simple (i.e. without internal degrees of freedom other from the spin) gases. In particular we will prove directly from our formalism that the second viscosity must vanish in the non-relativistic and in the ultra-relativistic limit (cf [58]). We will, then, present the equation of state of the gas, extended to quasi-equilibrium states, for the intermediate case, in a minimal model with $l = 2$.

⁷ In the literature [53, 56], Ξ is usually denoted by λ and equation (111) is expressed in the different notation $\Gamma = \lambda \delta \mu$.

8.1. Elements of kinetic theory

We need to specialise the analysis presented in section 2 to a gas in which interactions are given only by instantaneous collisions, in the limit of small cross sections. The single particle Hamiltonian is assumed invariant under spin flip, so there is full degeneracy in the spin. For clarity we present the derivation step by step.

Consider a homogeneous portion of the gas in a box with reflecting walls. The state of this gas is described with a distribution function $f = f(\mathbf{x}, \mathbf{p})$, which is the number of particles in per unit single-particle phase space volume,

$$f = \frac{dN}{d_3x d_3p}. \quad (147)$$

Since we work only in the frame identified by the box, we do not need to study the behaviour of f under Lorentz transformations. Because of homogeneity and isotropy, f depends only on the modulus of \mathbf{p} and describes a uniform local property of the matter elements. Note that if f were not isotropic the interaction of the particles with the walls could alter the value of f with time. However, since a collision sends a component $p_j \rightarrow -p_j$, an isotropic f is on average unaltered by this process.

We can use f to compute the particle density

$$n = \int f d_3p, \quad (148)$$

the energy density

$$\mathcal{U} = \int \epsilon f d_3p \quad (149)$$

and the isotropic stress

$$\Psi = \frac{1}{3} \int p_j v^j f d_3p, \quad (150)$$

where we have that $\epsilon = \epsilon(\mathbf{p})$ is the single-particle energy and

$$v^j = \frac{\partial \epsilon}{\partial p_j}. \quad (151)$$

It is also possible to compute the entropy per unit volume

$$s = \int \sigma f d_3p, \quad (152)$$

where

$$\sigma = -\ln\left(\frac{f h_p^3}{g}\right) + \left(1 - \frac{g \iota}{f h_p^3}\right) \ln\left(1 - \frac{\iota f h_p^3}{g}\right). \quad (153)$$

h_p is the Planck constant, g is a possible degeneracy of spin and ι is a coefficient which is equal to -1 for Bosons, to $+1$ for Fermions and to 0 in the classic limit.

Now we need to identify the two time-scales τ_m and τ_M . We define τ_m to be the typical time necessary for a particle to cross the box and τ_M to be inverse of the frequency of the binary collisions (which are assumed to be the dominant relaxation process). One may raise the

criticism that the hydrodynamic description can exist only if the collision frequency is larger than the characteristic hydrodynamic frequencies, implying $\tau_H \gg \tau_M$. This would lead to the claim that a hyperbolic hydrodynamic formulation for bulk viscosity in this case is not guaranteed to exist, but that only its parabolic and perfect-fluid limits are possible, see subsection 2.4. This serious issue is connected with the fact that l may *a priori* be infinite, as will be explained in more detail in subsection 9.5. In this section, however, we will assume that, for all practical applications, it is possible to approximate the system with a finite l model.

The assumption $\tau_m \ll \tau_M$ means that we are assuming that particles collide with the walls infinitely more often than with each other. During an expansion occurring in the time-scale τ_{fr} introduced in section 2 the particles do not have time to interact, but slam against the walls with an infinite frequency with respect to the rate of change of the position of the walls. In appendix C we show that if we parametrise the expansion with λ , such that $v \rightarrow \lambda^3 v$, then we have that in this process $f \rightarrow f_\lambda$, with

$$f_\lambda(\mathbf{p}) = f(\lambda\mathbf{p}). \quad (154)$$

We also prove that this transformation satisfies the conditions

$$\frac{dx_s}{d\lambda} = 0 \quad \frac{d\tilde{U}}{d\lambda} = -\Psi \frac{dv}{d\lambda}. \quad (155)$$

This is coherent with equations (8), (23) and (41).

8.2. Constructing the manifold of quasi-equilibrium states

Now we need to introduce the manifold \mathcal{Z} of quasi-equilibrium states. If f is a generic isotropic function, then the number l goes to infinity, because \mathcal{Z} should coincide with the Banach space \mathfrak{B} of the isotropic functions, which has infinite dimensions. To produce a conceivable hydrodynamic description directly from the kinetic theory one has to assume that only a limited subset of \mathfrak{B} is sufficient to describe the quasi-equilibrium states the fluid will explore. Here we show the general strategy to obtain this manifold.

First of all we note that the two-dimensional manifold of the local equilibrium macrostates has to be a submanifold of \mathcal{Z} . For a gas, the equilibrium distribution, i.e. the one which maximizes the entropy (152), fixed the density of particles and the energy density, and which will, therefore, satisfy (5) whichever set of α_A we choose, is

$$f_{\beta,\mu}(\mathbf{p}) = \frac{g}{h_p^3} \frac{1}{e^{\beta(\epsilon(\mathbf{p})-\mu)} + l}, \quad (156)$$

where β is the inverse of the temperature and μ is the chemical potential. So we need to impose $f_{\beta,\mu} \in \mathcal{Z}$.

Secondly, we note that the adiabatic expansion (154) must describe a group of transformations on \mathcal{Z} and must not send a point of \mathcal{Z} out of it. In mathematical terms, we may say that \mathcal{Z} must be invariant under the group of the adiabatic expansions, which send $f_{\beta,\mu}$ into

$$f_{\beta,\mu,\lambda}(\mathbf{p}) = \frac{g}{h_p^3} \frac{1}{e^{\beta(\epsilon(\lambda\mathbf{p})-\mu)} + l}. \quad (157)$$

So the set of the $f_{\beta,\mu,\lambda}$ must be a submanifold of \mathcal{Z} . Note that for $\lambda \neq 1$ it is not necessarily true that β and μ can be interpreted as inverse temperature and chemical potential. This is connected to the chemical gauge freedom discussed in subsection 2.5.

Finally, the manifold should be extended considering the collision processes. To understand how, imagine a homogeneous portion of a gas at rest in a box, prepared in an arbitrary out-of-equilibrium state. The collisions will tend to drive f towards equilibrium following a curve $f(t)$. If we want to be able to fully describe this curve in the framework of our hydrodynamic description we need $f(t)$ to be a map

$$f : \mathbb{R} \longrightarrow \mathcal{Z} \subset \mathfrak{B}. \tag{158}$$

On the other hand, the curve $f(t)$ is governed by an equation of the form

$$\frac{df}{dt} = \dot{f}_{\text{coll}}[f], \tag{159}$$

where the right-hand side is a collision functional. Therefore we need \mathcal{Z} to be an invariant set of the flux generated by \dot{f}_{coll} .

In practice, however, finding the submanifold of \mathfrak{B} which contains the equilibrium states and is invariant under both the adiabatic expansion and the flux generated by \dot{f}_{coll} can be hard, because the two generators do not commute,

$$\left[\frac{d}{d\lambda}, \dot{f}_{\text{coll}} \right] \neq 0, \tag{160}$$

where $d/d\lambda$ and \dot{f}_{coll} are seen as vector fields tangent to \mathfrak{B} . To prove the inequality (160), imagine to start in an equilibrium state: if we make a relaxation process followed by an adiabatic expansion, if the bulk viscosity is not zero the result is an out-of-equilibrium state. On the other hand, if we invert the order of the transformations and assume that the relaxation process is sufficiently long, we may end up in thermodynamic equilibrium.

In general, the dimension of \mathcal{Z} may be arbitrarily high (and we will explain the consequences of this problem in subsection 9.5) so it can be convenient to make an hierarchy of approximated theories with increasing l , starting with the minimal $l = 2$ manifold of $f_{\beta,\mu,\lambda}$ and expanding the set of included functions gradually, to increase the precision.

In this work we will restrict ourselves to the $l = 2$ model, which will allow us to have a direct comparison with the microphysical calculations of Israel and Stewart [20].

8.3. Proving that the second viscosity of non-relativistic and ultra-relativistic ideal gases vanishes

A non-relativistic gas is described through the dispersion law

$$\epsilon(\mathbf{p}) = m + \frac{\mathbf{p}^2}{2m}. \tag{161}$$

Plugging this condition inside (157) we find that

$$f_{\beta,\mu,\lambda} = f_{\beta',\mu'} \tag{162}$$

with

$$\beta' = \beta\lambda^2 \quad \mu' = m + \frac{\mu - m}{\lambda^2}. \tag{163}$$

An ultra-relativistic gas, on the other hand, is obtained imposing that the single-particle energy has the form

$$\epsilon(\mathbf{p}) = |\mathbf{p}|. \tag{164}$$

This, plugged in equation (157), gives again a relation of the type (162), with the transformation

$$\beta' = \lambda\beta \quad \mu' = \frac{\mu}{\lambda}. \quad (165)$$

Combining the first equations of (163) and (165) with the fact that $v' = \lambda^3 v$ we have verified that they follow the equilibrium adiabatic curves

$$\Theta' v'^{\Gamma-1} = \Theta v^{\Gamma-1}, \quad (166)$$

where $\Gamma = 5/3$ for the non-relativistic gas and $4/3$ for the ultra-relativistic gas. So we have proved that in both cases the adiabatic curves, generated in the fast expansion, which start in an equilibrium state, remain in the surface of the equilibrium states. This implies that if a fluid element is prepared in a local thermodynamic equilibrium state, then the expansions and contractions it will incur will not be able to drive it out of equilibrium, independently from the speed of the expansion/contraction, provided that $\tau_H \gg \tau_m$, proving that in these two cases the bulk viscosity vanishes, as explained also in [58, 59]. An alternative proof of this fact, based on geometrical arguments, is given in appendix A3.

8.4. The equation of state for a relativistic diluted massive gas out of equilibrium

Let us impose $\iota = 0$ and

$$\epsilon(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}. \quad (167)$$

Then the distribution in a generic state of \mathcal{Z} is

$$f_{\beta,\mu,\lambda} = \frac{g}{h_p^3} \exp \left[\alpha - \zeta_c \sqrt{1 + \frac{\lambda^2 \mathbf{p}^2}{m^2}} \right], \quad (168)$$

where we have introduced

$$\alpha = \beta\mu \quad \zeta_c = \beta m. \quad (169)$$

In the case $\lambda = 1$, when the fluid is in thermal equilibrium, α is the fugacity and ζ_c is the coldness [47], but out of equilibrium this interpretation is lost because β and μ are no longer the inverse temperature and chemical potential of the fluid. The variables $(\lambda, \alpha, \zeta_c)$ define a chart on \mathcal{Z} and the curves generated by W_{fr} are curves with constant α and ζ_c . This means that both α and ζ_c are good candidates to become the variable we can use to parametrise the quasi-equilibrium states.

We define the factor

$$b := \frac{4\pi g m^3}{h_p^3} \quad (170)$$

and the function

$$\phi(\zeta_c) := \ln \int_0^{+\infty} \xi^2 e^{-\zeta_c \sqrt{1+\xi^2}} d\xi, \quad (171)$$

whose first and second derivative in ζ_c will be denoted respectively by ϕ' and ϕ'' . Using equations (148) and (152), we find

$$v = \frac{\lambda^3}{b e^{\alpha+\phi}} \quad x_s = -\alpha - \zeta_c \phi'. \quad (172)$$

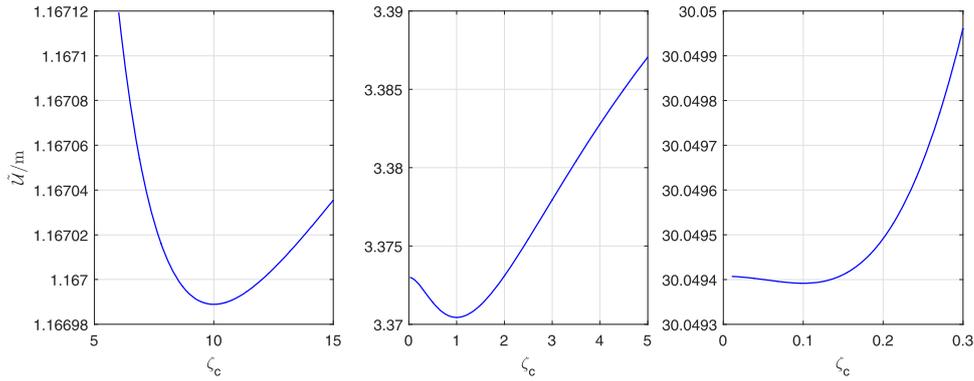


Figure 1. Plots of \tilde{U}/m as a function of ζ_c , with v and x_s fixed and chosen in a way that: $\zeta_c^{\text{eq}} = 10$ (first panel), $\zeta_c^{\text{eq}} = 1$ (second panel), $\zeta_c^{\text{eq}} = 0.1$ (third panel).

Note that x_s does not depend on λ and this is coherent with the fact that it is conserved along the curves generated by W_{fr} . We are, now, able to introduce the chart (v, x_s, ζ_c) , which is in the form discussed in section 2.2. We can invert the foregoing equations obtaining

$$\alpha = -x_s - \zeta_c \phi' \quad \lambda = \left(b v e^{\phi - \zeta_c \phi' - x_s} \right)^{1/3}. \tag{173}$$

The distribution f is naturally given in terms of the parameters λ , α and ζ_c , so it is natural to use these variables while performing an average to extract a thermodynamic variable. Then, making the coordinate transformation

$$(\lambda, \alpha, \zeta_c) \mapsto (v, x_s, \zeta_c) \tag{174}$$

by means of the formulas (173), the quantity can be finally written as a function of multifluid-type variables. In particular this can be done to obtain the equation of state $\tilde{U}(v, x_s, \zeta_c)$. In fact, recalling equation (149) and making use of the first equation of (172), we find

$$\tilde{U}(\lambda, \zeta_c) = m e^{-\phi} \int_0^{+\infty} \xi^2 e^{-\zeta_c \sqrt{1+\xi^2}} \sqrt{1 + \frac{\xi^2}{\lambda^2}} d\xi. \tag{175}$$

We remark that the fact that \tilde{U} does not depend on α is a useful result which holds only in the non-degenerate limit. Making the change of variables we finally obtain the equation of state for the quasi-equilibrium gas:

$$\tilde{U}(v, x_s, \zeta_c) = m e^{-\phi} \int_0^{+\infty} \xi^2 e^{-\zeta_c \sqrt{1+\xi^2}} \sqrt{1 + \xi^2 \left(\frac{e^{x_s - \phi + \zeta_c \phi'}}{b v} \right)^{2/3}} d\xi. \tag{176}$$

In figure 1 it is possible to see three plots representing \tilde{U}/m as a function of ζ_c . v and x_s are held fixed and chosen is such a way that when $\lambda = 1$, i.e. in thermal equilibrium, ζ_c takes the values respectively 10 (first panel), 1 (second panel) and 0.1 (third panel). The energy has its absolute minimum in the equilibrium state, coherently with the results of section 4.1.

8.5. Pressure, temperature and affinity in the quasi-equilibrium states

To compute the derivatives of the energy per particle it is convenient to pass through the variable λ . For example, to compute the pressure, according to (10) and (41), we only need to calculate

$$\Psi = -\left.\frac{\partial \tilde{\mathcal{U}}}{\partial v}\right|_{x_s, \zeta_c} = -\left.\frac{\partial \tilde{\mathcal{U}}}{\partial \lambda}\right|_{\zeta_c} \left.\frac{\partial \lambda}{\partial v}\right|_{x_s, \zeta_c}. \tag{177}$$

Using (173), we see that

$$\left.\frac{\partial \lambda}{\partial v}\right|_{x_s, \zeta_c} = \frac{\lambda}{3v}, \tag{178}$$

and we get the formula

$$\Psi = \frac{m e^{-\phi}}{3v} \int_0^{+\infty} e^{-\zeta_c \sqrt{1+\xi^2}} \frac{\xi^4/\lambda^2}{\sqrt{1+\xi^2/\lambda^2}} d\xi. \tag{179}$$

It is possible to check with a little algebra that this coincides with (150), proving the consistency of the formulation.

The temperature, according to (10), is

$$\Theta = \left.\frac{\partial \tilde{\mathcal{U}}}{\partial x_s}\right|_{v, \zeta_c} = \left.\frac{\partial \tilde{\mathcal{U}}}{\partial \lambda}\right|_{\zeta_c} \left.\frac{\partial \lambda}{\partial x_s}\right|_{v, \zeta_c}. \tag{180}$$

However, it is true that

$$\left.\frac{\partial \lambda}{\partial x_s}\right|_{v, \zeta_c} = -\frac{\lambda}{3}, \tag{181}$$

therefore, combining with (177) and (178) we find the equation

$$\Psi = n\Theta. \tag{182}$$

Thus we have found that the ideal gas law, which must be verified in equilibrium, remains true also out of it, provided that we choose ζ_c to be the additional variable in the equation of state.

The variable \mathbb{A} is calculated in appendix A4. We also verify there that in equilibrium (when $\lambda = 1$) it vanishes, in agreement with the fact that this is the condition of thermodynamic equilibrium. This is also in accordance with the plots of figure 1.

8.6. The Israel–Stewart limit of the theory

The full equation of state can be expanded for small deviations from equilibrium. As shown in section 7, this will lead us to Israel–Stewart theory for bulk viscosity.

Let us define the functions

$$J(\zeta_c) := \frac{e^{-\phi}}{3} \int_0^{+\infty} e^{-\zeta_c \sqrt{1+\xi^2}} \frac{\xi^4}{(1+\xi^2)^{3/2}} d\xi \tag{183}$$

and

$$G(\zeta_c) := \frac{K_3(\zeta_c)}{K_2(\zeta_c)}, \tag{184}$$

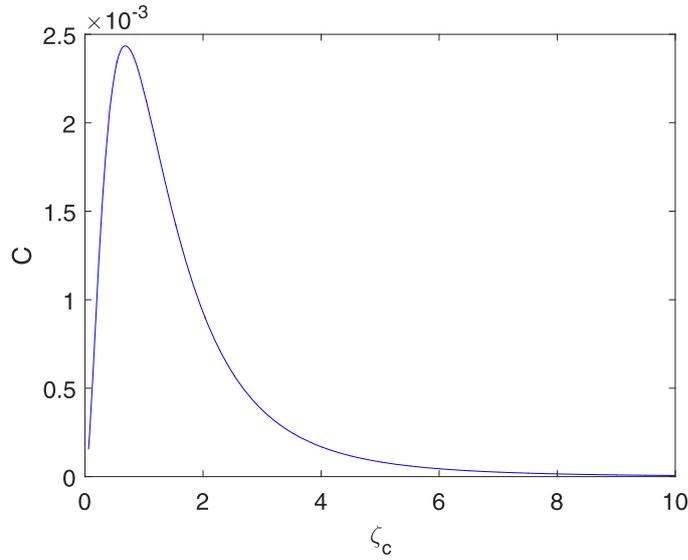


Figure 2. Plot of the susceptibility \mathcal{C} as a function of ζ_c .

where K_ν is the ν th modified Bessel function of the second type. Then it can be shown (see [47] and references therein) that

$$\begin{aligned} \phi &= \ln\left(\frac{K_2}{\zeta_c}\right) \\ \phi' &= \frac{1}{\zeta_c} - G \\ \phi'' &= -\frac{1}{\zeta_c^2} - G^2 + \frac{5G}{\zeta_c} + 1. \end{aligned} \tag{185}$$

In appendix A5 we prove that

$$\left. \frac{\partial^2 \tilde{\mathcal{U}}}{\partial \zeta_c^2} \right|_{v, x_s, \zeta_c = \zeta_c^{\text{eq}}} = \frac{m \zeta_c \phi''^2}{3} \left(1 + \zeta_c J - \frac{3}{\zeta_c^2 \phi''} \right). \tag{186}$$

The function

$$\mathcal{C} := \frac{1}{\tilde{\mathcal{U}}_{\text{eq}}} \left. \frac{\partial^2 \tilde{\mathcal{U}}}{\partial \zeta_c^2} \right|_{v, x_s, \zeta_c = \zeta_c^{\text{eq}}} \tag{187}$$

describes the susceptibility of the fluid to a displacement from equilibrium. In figure 2 it is possible to see the behaviour of \mathcal{C} as ζ_c varies. In the limits $\zeta_c \rightarrow 0$ (ultra-relativistic limit) and $\zeta_c \rightarrow +\infty$ (non-relativistic limit) it goes to zero, while it has its maximum around $\zeta_c = 1$. This is a result of the fact that in the two opposite limits the system becomes degenerate in ζ_c and the equation of state depends only on two independent variables: $\tilde{\mathcal{U}}(v, x_s, \zeta_c) = \tilde{\mathcal{U}}(v, x_s)$, cf with section 8.3.

We can finally compute the thermodynamic coefficient χ introduced in equation (131) and compare it with the prediction of [20].

Since equation (176) cannot be easily inverted to write the entropy as a function of $v, \tilde{\mathcal{U}}$ and ζ_c , it is more convenient to recast (131) in a form which involves derivatives of the energy per particle. Equation (132) can be easily used to prove that

$$\frac{y_s''}{\Theta} = \left(\frac{\partial^2 \tilde{\mathcal{U}}}{\partial \zeta_c^2} \Big|_{v, x_s} \right)^{-1}, \tag{188}$$

where the second derivative is evaluated in equilibrium. This, plugged in (131), gives

$$\chi = \frac{n}{\mathcal{UC}} \left(\frac{\partial \zeta_c^{\text{eq}}}{\partial v} \Big|_{x_s} \right)^{-2}. \tag{189}$$

Since in equilibrium $\lambda = 1$, taking the logarithm of the second equation of (173) we find that $\zeta_c^{\text{eq}}(v, x_s)$ has to satisfy the condition

$$\ln b + \phi(\zeta_c^{\text{eq}}) + \ln v - x_s - \zeta_c^{\text{eq}} \phi'(\zeta_c^{\text{eq}}) = 0. \tag{190}$$

Deriving with respect to v , keeping x_s fixed, we obtain

$$\frac{\partial \zeta_c^{\text{eq}}}{\partial v} \Big|_{x_s} = \frac{1}{v \zeta_c^{\text{eq}} \phi''(\zeta_c^{\text{eq}})}. \tag{191}$$

So we finally find

$$\chi = \frac{3}{P_{\text{eq}}} \left(1 + \zeta_c J - \frac{3}{\zeta_c^2 \phi''} \right)^{-1}. \tag{192}$$

This can be compared with the prediction of Israel and Stewart [20] (based on the Grad 14-moment approximation)

$$\chi_{\text{IS}} = \frac{3}{P_{\text{eq}}} \frac{5 - 3\gamma + 3(10 - 7\gamma)G/\zeta_c}{G^2(3\gamma - 5 + 3\gamma/G\zeta_c)^2}, \tag{193}$$

where γ is defined through the equation

$$\frac{\gamma}{\gamma - 1} = \zeta_c^2 \left(1 + \frac{5G}{\zeta_c} - G^2 \right). \tag{194}$$

In the first panel of figure 3 we can see the comparison between the two predictions for $mn\chi$. In the second panel we show their ratio. We compare $mn\chi$ because they are dimensionless and depend only on ζ_c . As can be seen from the figure, Israel–Stewart’s prediction is always larger than the one of the quasi-equilibrium equation of state, but they become equal for low temperatures.

8.7. Comparison

In this subsection we compare our kinetic approach with the Grad 14-moment approximation of Israel and Stewart [20], with the aim of explaining the behaviour of the second panel of figure 3.

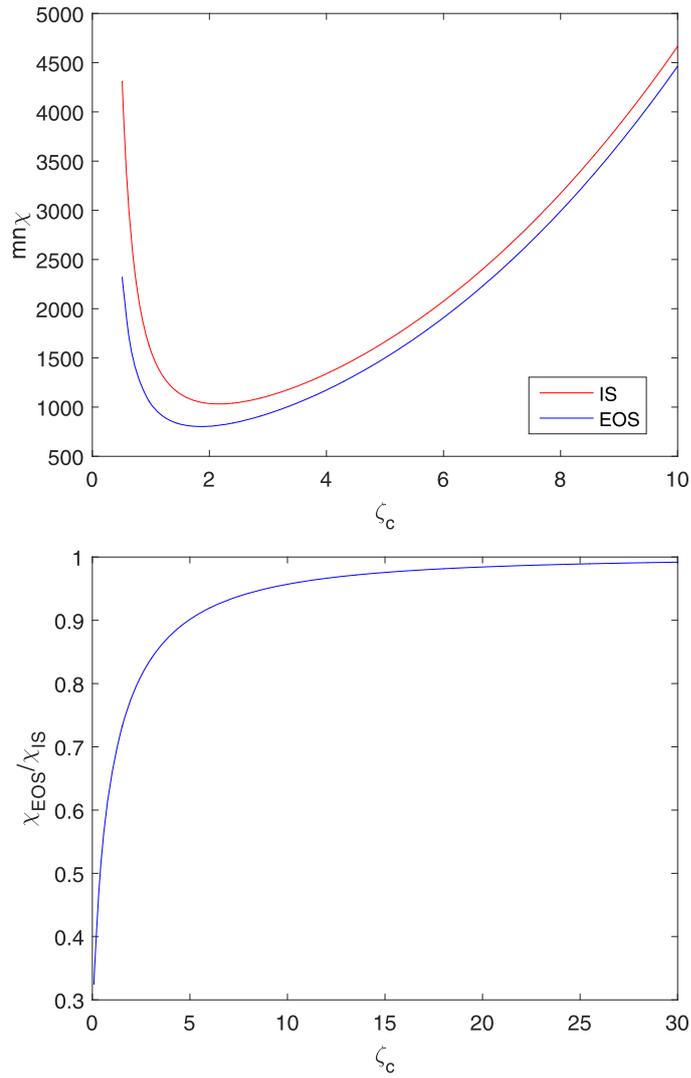


Figure 3. Upper panel: plot of the factor $mn\chi$ as a function of ζ_c according to the quasi-equilibrium equation of state (176) (blue line) and to Israel and Stewart (red line). Lower panel: plot of the ratio between the two. We see that when the gas becomes relativistic ($\Theta \gtrsim m$) the convexity coefficient χ calculated in Israel and Stewart [20] diverges from the results obtained with the quasi-equilibrium equation of state.

Following the steps of [20], we focus on the function σ we introduced in equation (153), which for $\iota = 0$ (non-degenerate limit) becomes

$$\sigma = -\ln\left(\frac{fh_p^3}{g}\right) \tag{195}$$

and, using (168), is equal to

$$\sigma = -\alpha + \zeta_c \sqrt{1 + \frac{\lambda^2 \mathbf{p}^2}{m^2}}. \quad (196)$$

Since our aim is to study f near equilibrium, we impose

$$\lambda = 1 + \delta\lambda \quad (197)$$

and we expand in $\delta\lambda$. If we define

$$\sigma_0 := -\alpha + \zeta_c \sqrt{1 + \frac{\mathbf{p}^2}{m^2}}, \quad (198)$$

we find that

$$\sigma = \sigma_0 + \frac{\zeta_c \mathbf{p}^2}{m\epsilon} \delta\lambda. \quad (199)$$

On the other hand [20], postulate a dependence of σ on \mathbf{p} (for locally isotropic matter elements) of the form

$$\sigma_{\text{IS}} = \sigma_0 + \mathbf{p}^2 \nu, \quad (200)$$

where ν is a parameter describing the displacement from equilibrium. For $\zeta_c \gg 1$ only the low energy states are significantly explored, $\epsilon \approx m$, therefore equation (199) reduces to (200) with

$$\nu = \frac{\zeta_c \delta\lambda}{m^2}. \quad (201)$$

Therefore we have shown that in the low temperature limit the two approaches coincide and this is reflected in the fact that the respective predictions for χ tend to be the same for $\zeta_c \rightarrow +\infty$. On the other hand, when the fluid becomes relativistic, the assumption about the shape of the perturbation becomes relevant. Considering that for large $|\mathbf{p}|$ we observe two different asymptotic behaviours:

$$\sigma \sim \frac{\zeta_c \lambda}{m} |\mathbf{p}| \quad \sigma_{\text{IS}} \sim \nu \mathbf{p}^2, \quad (202)$$

as the temperature increases we expect the accordance to fail, as it can be seen in figure 3. Substantially, our formulation models directly the evolution of the particle momentum distribution, which in the high temperature limit differs from the ansatz of [20].

8.8. The role of degeneracy

The calculations in subsections 8.4–8.7 can be performed also for a degenerate Fermi gas. However, since in the Fermi–Dirac case α cannot be factorised out in the integral expression for $n(\alpha, \zeta_c, \lambda)$, there seems to be no way of inverting this relation. This prevents us from approaching the problem analytically. Hence, we will present here only the plot of $mn\chi$ as a function of the chemical potential, see figure 4.

If $\mu < m$ we are in the non-degenerate limit and we recover the results of the previous subsections. In particular, in this limit $mn\chi$ does not depend on μ , so we obtain the plateau that can be seen in figure 4 for low μ . When $\mu > m$, however, $mn\chi \rightarrow +\infty$. To understand this, consider the definition of the susceptibility given in equation (187). In the variations performed to compute the second derivative one has to change only the shape of the distribution, keeping the same density of particles and entropy. However, in a degenerate gas, $f h_p^3/g$ is

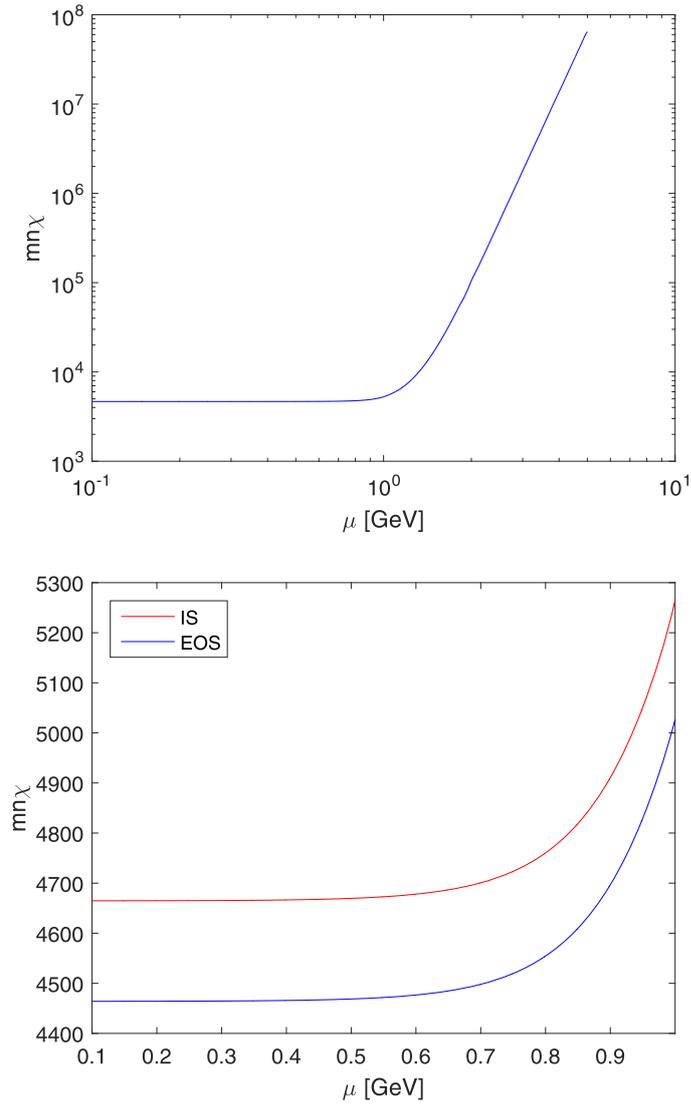


Figure 4. Plot of $mn\chi$ as a function of the relativistic chemical potential. We are considering a gas of particles of mass 1 GeV, with a temperature $\Theta = 10^{12}$ K.

everywhere 1 or 0, apart from a thin shell of momenta around the Fermi surface, whose thickness is proportional to the temperature. So only a small fraction of electrons near the Fermi momentum is involved in the variation. Since we are dividing by the whole energy per particles, $\mathcal{C} \rightarrow 0$. As a result, $mn\chi \rightarrow +\infty$, see equation (189).

In the degenerate limit the ratio $\chi_{\text{EOS}}/\chi_{\text{IS}} \rightarrow 1$, which can be explained with a simple argument. Following [20], we introduce

$$\eta = \ln \left(\frac{f h_p^3/g}{1 - f h_p^3/g} \right). \quad (203)$$

It is possible to check that in our theory

$$\eta = \alpha - \zeta_c \sqrt{1 + \frac{\lambda^2 p^2}{m^2}}. \quad (204)$$

Analogously to what we did in the previous subsection we note that for small displacements from equilibrium we have

$$\eta = \alpha - \zeta_c \sqrt{1 + \frac{p^2}{m^2}} - \frac{\zeta_c p^2}{m\epsilon} \delta\lambda. \quad (205)$$

However in a degenerate gas the momenta which are involved moving out of equilibrium are a thin shell near the Fermi surface, so we can impose $p = p_F + q$, with q small. So we can expand the last term and find

$$\eta = \alpha - \frac{\zeta_c p_F^2}{m\epsilon_F} \delta\lambda - \zeta_c \sqrt{1 + \frac{p^2}{m^2}} - \frac{\zeta_c \delta\lambda}{m} \frac{d}{dp} \left(\frac{p^2}{\epsilon} \right) \Big|_{p_F} q. \quad (206)$$

Defining

$$\alpha_P := \alpha - \frac{\zeta_c p_F^2}{m\epsilon_F} \delta\lambda \quad \nu_P := - \frac{\zeta_c \delta\lambda}{m} \frac{d}{dp} \left(\frac{p^2}{\epsilon} \right) \Big|_{p_F}, \quad (207)$$

we arrive at the form

$$\eta = \alpha_P - \zeta_c \sqrt{1 + \frac{p^2}{m^2}} + \nu_P q. \quad (208)$$

On the other hand [20], assume a near-equilibrium distribution function of the form

$$\eta_{IS} = \alpha_{IS} - \zeta_c \sqrt{1 + \frac{p^2}{m^2}} + \nu p^2. \quad (209)$$

Making the same expansion we obtain

$$\eta_{IS} = \alpha_{IS} + \nu p_F^2 - \zeta_c \sqrt{1 + \frac{p^2}{m^2}} + 2\nu p_F q. \quad (210)$$

Then, noting that we can define

$$\alpha_P := \alpha_{IS} + \nu p_F^2 \quad \nu_P := 2\nu p_F, \quad (211)$$

we recover (208). Considering that far from the Fermi surface $f h_p^3/g$ can be approximated to 0 or 1, we have shown that the prescription for the shape of the distribution function near equilibrium, in the degenerate limit, presented in [20] coincides with ours. The only difference between the two approaches is given by the fact that we are using different charts, but the manifold \mathcal{Z} is the same. Since all the physical quantities are independent from the initial chart which is employed in kinetic theory, the two theories are the same and $\chi_{EOS}/\chi_{IS} \rightarrow 1$.

In figure 5 we show a plot of the behaviour of the ratio between the predictions of χ according to our equation of state and of [20] for particles having the mass of the nucleons. To emphasize possible differences, the plot refers to the non-degenerate limit.

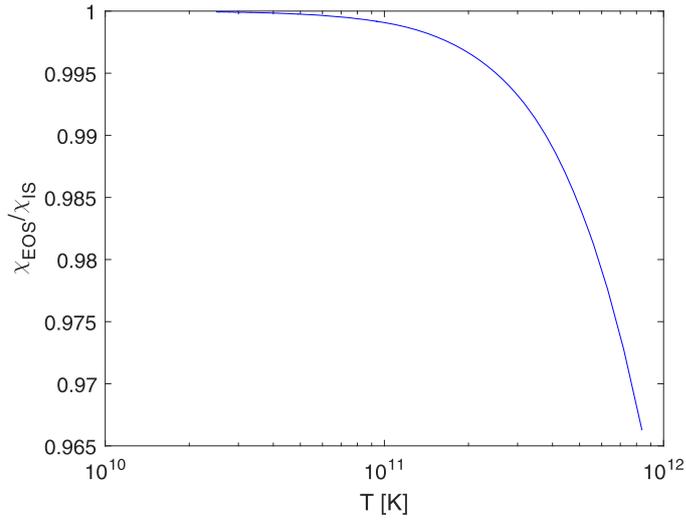


Figure 5. Plot of the ratio $\chi_{\text{EOS}}/\chi_{\text{IS}}$ as a function of temperature for a Fermion gas of particles with mass $m = 1$ GeV. We consider the case $\mu \ll m$ (the plateau in figure 4), because as the degeneration increases the function goes to 1.

9. From transport equations to bulk viscosity

In the previous section we have shown how a kinetic description of an ideal gas can be used to compute the quasi-equilibrium equation of state. It is, however, possible to start directly from a relativistic transport equation and prove that the hydrodynamic description presented in section 3 can be derived from it. We report here the steps of this proof as a final demonstration of the coherence and universality of the approach. In addition, it will be evident that, at the level of kinetic theory, our approach is the direct result of a change of variables (the straightification of W_{fr}) in the formalism of Israel–Stewart, as can be verified through a direct comparison with [20].

In order to simplify the equations we will work in the context of special relativity in Minkowskian coordinates.

9.1. The continuity equations of kinetic theory

The evolution equation of the single-particle distribution (assuming absence of external forces) in a flat spacetime is [60]

$$p^\nu \partial_\nu f = \mathfrak{C}[f], \tag{212}$$

where \mathfrak{C} is a collision term. Let us introduce the transport fluxes

$$\varphi_{(N)}^{\nu\alpha_1 \dots \alpha_N} := \int p^{\alpha_1} \dots p^{\alpha_N} p^\nu f \frac{d_3 p}{p^0} \tag{213}$$

and the collision tensors

$$\mathfrak{C}_{(N)}^{\alpha_1 \dots \alpha_N} := \int p^{\alpha_1} \dots p^{\alpha_N} \mathfrak{C} \frac{d_3 p}{p^0}. \tag{214}$$

They are totally symmetric tensors and, using (212), satisfy

$$\partial_\nu \varphi_{(N)}^{\nu\alpha_1 \dots \alpha_N} = \mathfrak{C}_{(N)}^{\alpha_1 \dots \alpha_N}. \tag{215}$$

Note that $\varphi_{(0)}^\nu = n^\nu$ and $\varphi_{(1)}^{\nu\rho} = T^{\nu\rho}$ [60]. Conservation of particles and energy–momentum in the collisions imply

$$\mathfrak{C}_{(0)} = 0 \quad \mathfrak{C}_{(1)}^{\alpha_1} = 0 \tag{216}$$

and (215) for $N = 0$ and $N = 1$ respectively become

$$\partial_\nu n^\nu = 0 \quad \partial_\nu T^{\nu\rho} = 0, \tag{217}$$

in accordance with (32) and (34). We can, now, introduce the tensors

$$\begin{aligned} n_{(N)}^\nu &:= (-1)^N \varphi_{(N)}^{\nu\alpha_1 \dots \alpha_N} u_{\alpha_1} \dots u_{\alpha_N} \\ T_{(N)}^{\nu\rho} &:= (-1)^{N-1} \varphi_{(N)}^{\nu\rho\alpha_2 \dots \alpha_N} u_{\alpha_2} \dots u_{\alpha_N}. \end{aligned} \tag{218}$$

It can be seen that

$$n_{(0)}^\nu = n^\nu \quad T_{(1)}^{\nu\rho} = T^{\nu\rho}. \tag{219}$$

Since $\varphi_{(0)}^\nu$ has only one index, $T_{(0)}^{\nu\rho}$ is not well defined, but we extend the definition imposing

$$T_{(0)}^{\nu\rho} := 0. \tag{220}$$

Applying the divergence to the first equation of (218) we obtain the continuity equations

$$\partial_\nu n_{(N)}^\nu = nc_N - NT_{(N)}^{\nu\rho} \partial_\nu u_\rho, \tag{221}$$

where we have introduced the scalars

$$c_N = \frac{(-1)^N}{n} \mathfrak{C}_{(N)}^{\alpha_1 \dots \alpha_N} u_{\alpha_1} \dots u_{\alpha_N}. \tag{222}$$

9.2. The local isotropy assumption

Now we need to impose the condition that bulk viscosity is the only dissipative process occurring in the system. This, in section 3, was encoded in the requirement of local isotropy in the reference frame of the fluid element. Therefore, in accordance with that approach we assume that in each point of the spacetime, f is invariant under rotations in the reference frame defined by n^ν . Then, it is immediate to show that, defined (from now on we work with $N > 0$)

$$\begin{aligned} \tilde{\mathcal{U}}_N &:= \frac{(-1)^{N+1}}{n} \varphi^{\nu\alpha_1 \dots \alpha_N} u_\nu u_{\alpha_1} \dots u_{\alpha_N} \\ \Psi_N &:= [T_{(N)}^{\nu\rho} g_{\nu\rho} + n\tilde{\mathcal{U}}_N]/3, \end{aligned} \tag{223}$$

the tensors introduced in (218) have the form

$$\begin{aligned} n_{(N)}^\nu &= \tilde{\mathcal{U}}_N n^\nu \\ T_{(N)}^{\nu\rho} &= [n\tilde{\mathcal{U}}_N + \Psi_N] u^\nu u^\rho + \Psi_N g^{\nu\rho}. \end{aligned} \tag{224}$$

This plugged into the continuity equations, we find

$$\dot{\tilde{U}}_N = c_N - N\Psi_N\dot{v}. \quad (225)$$

For $N = 1$ we obtain (38), since $\tilde{U}_{(1)} = \tilde{U}$ and $\Psi_{(1)} = \Psi$. The foregoing equation describes the evolution of the internal macroscopic degree of freedom \tilde{U}_N , which is the particle average value of $(p^0)^N$ computed in the reference frame of the fluid. There are two contributions in the right-hand side: a collisional term, which represents the dissipative processes which tend to lead the system to local thermal equilibrium, and the direct effect of the expansion of the volume element.

9.3. The thermodynamics of the fluid

It can be shown, see appendix A6, that the infinite set of the \tilde{U}_N , together with n , contains the same amount of information as f . On the other hand, in order for a hydrodynamical description of the fluid to be possible, we need to assume that only a finite number of them is necessary to know the macroscopic local state of matter. Therefore we can impose that

$$(v, \tilde{U}_N)_{N=1, \dots, l} \quad (226)$$

is a global chart of the manifold of the thermodynamic states. This is equivalent to saying that all the \tilde{U}_N for $N > l$ can be written as functions of v and the \tilde{U}_N with $N \leq l$. As l grows we explore an hierarchy of more and more refined fluid theories. For $l = 0$ we have the barotropic perfect fluid, for $l = 1$ the finite temperature perfect fluid, for $l > 1$ we have more and more complicated models of bulk viscosity and, in the limit $l \rightarrow +\infty$ we recover full kinetic theory.

We have written the manifold of states in the form (3), where the \tilde{U}_N for $N > 1$ play the role of the α_A . Now we need to introduce the vector field W_{fr} . In the chart (226) it can be decomposed as

$$W_{\text{fr}} = \frac{\partial}{\partial v} + \sum_{N=1}^l W_{\text{fr}}(\tilde{U}_N) \frac{\partial}{\partial \tilde{U}_N}. \quad (227)$$

Its action on a thermodynamic variable gives the derivative of the variable with respect to the volume per particle along the curve drawn by the system in an expansion which is faster than the relaxation time-scale, see subsection 2.2. On the other hand, we show in appendix C that the aforementioned curve can be described in a parametric way, in the context of kinetic theory of weakly interacting gases, through the condition (C21).⁸ Therefore, using the chain rule through equation (C22), we can write

$$W_{\text{fr}}(\tilde{U}_N) = \left. \frac{1}{3v} \frac{d\tilde{U}_N(\lambda)}{d\lambda} \right|_{\lambda=1}. \quad (228)$$

Using the first definition in (223) and the fact that we are in the reference frame of the matter element, we find

$$W_{\text{fr}}(\tilde{U}_N) = -\frac{N}{3} \int (m^2 + \mathbf{p}^2)^{N/2-1} \mathbf{p}^2 f(\mathbf{p}) d_3 p. \quad (229)$$

⁸ We are working in the reference frame of the fluid element here.

On the other hand it is possible to verify that

$$\Psi_N = \frac{1}{3} \int (m^2 + \mathbf{p}^2)^{N/2-1} \mathbf{p}^2 f(\mathbf{p}) d_3 p, \tag{230}$$

hence we obtain

$$W_{\text{fr}}(\tilde{\mathcal{U}}_N) = -N\Psi_N. \tag{231}$$

Note that, remembering equation (23), the above formula reduces to (41) for $N = 1$.

Plugging (231) into (225) we obtain

$$\dot{\tilde{\mathcal{U}}}_N = c_N + W_{\text{fr}}(\tilde{\mathcal{U}}_N)\dot{v}. \tag{232}$$

If we impose the collision term to be zero, the above equation becomes

$$\left. \frac{d\tilde{\mathcal{U}}_N}{dv} \right|_{\text{along a worldline}} = W_{\text{fr}}(\tilde{\mathcal{U}}_N), \tag{233}$$

which means that in the absence of dissipation the thermodynamic variables change along the worldlines according to the transformation (C21), produced by the expansions and contractions of the volume element.

9.4. The chemical-like chart

Since the chart (226) has the form (3) and we have the vector field W_{fr} , we can introduce a chart of the type (v, x_s, x_A) ,⁹ see subsection 2.2, with $x_A = x_A(v, \tilde{\mathcal{U}}_N)$ dimensionless variables satisfying the condition

$$W_{\text{fr}}(x_A) = 0. \tag{234}$$

Using the expansion (227), we find

$$\frac{\partial x_A}{\partial v} + \sum_{N=1}^l W_{\text{fr}}(\tilde{\mathcal{U}}_N) \frac{\partial x_A}{\partial \tilde{\mathcal{U}}_N} = 0, \tag{235}$$

where the partial derivatives are referred to the chart (226).

We can compute the variation of x_A along the worldlines of the matter elements. Using the chain rule we can write

$$\dot{x}_A = \frac{\partial x_A}{\partial v} \dot{v} + \sum_{N=1}^l \frac{\partial x_A}{\partial \tilde{\mathcal{U}}_N} \dot{\tilde{\mathcal{U}}}_N, \tag{236}$$

with the aid of equations (232) and (235) we obtain

$$\dot{x}_A = c_A, \tag{237}$$

with

$$c_A = \sum_{N=1}^l \frac{\partial x_A}{\partial \tilde{\mathcal{U}}_N} c_N. \tag{238}$$

⁹Recall that $W_{\text{fr}}(x_s) = 0$ is guaranteed to be true because of equation (C28), see appendix C for more details. Furthermore, equation $\dot{x}_s \geq 0$ is a necessary product of Boltzmann's H-theorem [60].

In the absence of collisions (dissipation), c_A vanishes, so x_A is constant along the worldline, recovering the frozen limit. This proves, directly from kinetic theory, since it is a consequence of the continuity equation (221), that it is always possible to find l coordinates out of $l + 1$ thermodynamic degrees of freedom which are altered only by the relaxation processes.

The final step we need to make consists of showing that, near equilibrium, equation (237) can be rewritten in the form (48). However this is not a hard task, in fact we can write the c_A in the chart (v, x_s, \mathbb{A}^A) and expand near equilibrium (for small \mathbb{A}^A):

$$c_A(v, x_s, \mathbb{A}^B) = c_A(v, x_s, 0) + \frac{\partial c_A}{\partial \mathbb{A}^B}(v, x_s, 0)\mathbb{A}^B. \quad (239)$$

Since the collision term \mathfrak{C} vanishes in equilibrium we need to impose $c_A(v, x_s, 0) = 0$. Defining

$$\Xi_{AB} = n \frac{\partial c_A}{\partial \mathbb{A}^B}(v, x_s, 0), \quad (240)$$

we conclude the proof.

9.5. The explosion of the degrees of freedom and the universality of the Navier–Stokes equation

At the beginning of section 9.3 we assumed that only a finite number of $\tilde{\mathcal{U}}_N$ were independent. This is a necessary condition to make the hydrodynamic description possible and is one of the fundamental assumptions invoked in sections 2–4. Although this may be a good approximation already for $l = 2$ (depending on the shape of the collision integral), this may not be always the case: we cannot exclude *a priori* that a large amount of variables are required to describe the fluid. The condition $l \rightarrow +\infty$ implies that there are infinite telegraph-type equation (79), making the hydrodynamic problem unsolvable. This pathological explosion of degrees of freedom is a symptom of the issue pointed out in subsection 8.1: in this system the existence of a hydrodynamic description is guaranteed only for $\tau_H \gg \tau_M$.

This problem marks a fundamental difference between the hyperbolic and the parabolic case. As pointed out in section 4.3, in the hyperbolic case a large l implies more independent variables and, therefore, an increasing complexity. On the other hand, in the parabolic limit all the contributions add up, see equation (84), in the coefficient ζ which, alone, contains all the information about the thermodynamics and kinetics of the substance. For this reason, even if $l \rightarrow +\infty$, in the parabolic limit the hydrodynamic theory is still possible, making Navier–Stokes a universal equation. Thus, a weakly interacting gas, whose bulk viscosity is produced by the relaxation of the momentum distribution to equilibrium, is a perfect example of a fluid which is efficiently described through Navier–Stokes, but which in the hyperbolic regime may need an l larger than 2, in contrast with what has been done up to now, as discussed in subsection 2.1.

10. Conclusions

We derived, by application of the principles of extended irreversible thermodynamics, a general hydrodynamic description for bulk-viscous fluids in general relativity, that builds on the covariant multifluid formalism of Carter and collaborators [26, 61, 62] and naturally allows for a symmetric hyperbolic form of the system [32] and causal solutions. The present theory is thus well suited for numerical applications.

Our model is not based on near-equilibrium assumptions, but only a time-scale separation between the equilibration of a restricted (tractable) number of macroscopic degrees of freedom

and all the microscopic ones. We have proven that this can always be recast into a multi-component single fluid with a number of currents which grows with the number of out-of-equilibrium degrees of freedom.

In this context, dissipation is modelled as chemical transfusion, even in the absence of real chemical reactions. The chemically-induced bulk viscosity in neutron stars has been shown to be the simplest particular case of our description, and the dependence of the effective bulk viscosity coefficient on the frequency, generally derived from a perturbative approach [53], has been shown to arise directly from the telegraph-type form equations in our model. Therefore our approach provides the machinery to take any equation of state and set of reaction rates (e.g. in the neutron star interior) and construct a hyperbolic model for bulk viscosity. In such a context the standard bulk viscosity coefficient itself can also be recovered by taking the parabolic (low frequency) limit of the theory.

The model (for bulk viscosity) of Israel and Stewart [20] has been proven to emerge as an expansion near equilibrium of our general approach, and we give a formula for its coefficients in the chemically-induced case. Varying the dynamical time-scale of the hydrodynamics, different regimes of the theory can appear, such as the fast limit in which the new degrees of freedom appear to be frozen on the hydrodynamic time-scale, and the slow limit in which the parabolic Navier–Stokes equations are recovered.

The approach has been, finally, employed in the context of the kinetic theory of ideal gases of fermions. This has provided a proof of the fact that bulk viscosity vanishes in the non-relativistic and in the ultra-relativistic limit. In the intermediate case we have derived the equation of state for the out-of-equilibrium gas in a minimal model with $l = 2$ (i.e. allowing for the equation of state to depend on temperature and an additional variable), and calculated the bulk viscosity coefficients. In the degenerate and cold limit our results coincide with those derived from the Grad 14-moment approximation of [20]. At higher temperatures, for $T \gtrsim 1$ MeV (i.e. $T \gtrsim 10^{10}$ K) however, while the behaviour is qualitatively similar, there are quantitative differences between the two approaches, as our formulation models directly the evolution of the particle momentum distribution, which in the high temperature limit differs from the ansatz of [20].

The results of this paper are applicable to a number of astrophysical problems in which a general relativistic description of viscous matter is needed, including numerical studies of a binary neutron star merger and its remnant, but also for the damping of modes of oscillation of compact as well as classical stars. Indeed, we would like to remark that they have recently been used in the context of radiation hydrodynamics, to model the dissipation which is generated from the interaction between matter and radiation in the limit of infinite elastic-scattering opacity [13].

Future directions of investigation concern the possibility of relaxing the assumption of isotropy of the fluid element. This naturally leads one to the inclusion of heat conduction, charge conductivity and shear viscosity in the theory. In addition, relaxing the local rotation-invariance assumption is a necessary step for a consistent implementation of superfluidity. Unfortunately, the straightification technique used in subsection 2.2 to simplify the formulation of the dynamics cannot be applied to anisotropic fluid elements, as expansions in orthogonal directions may not commute with each other (preventing a simultaneous straightification). As a consequence, within the extended irreversible thermodynamics framework discussed here, there is a large variety of different models for shear viscosity that have the same number of degrees of freedom [6, 22, 63], contrarily to what happens for pure bulk viscosity.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Mathematical calculations

In this appendix we present the proofs which were removed from the main body of the paper.

A1. Straightifying the generator of the fast expansions

Let us choose an arbitrary reference volume v_0 and define \mathcal{Z}_{v_0} to be the l -dimensional submanifold of \mathcal{Z} given by the condition $v = v_0$. It represents the set of all the possible quasi-equilibrium states in which the fluid can be found when the opposite walls of the box have a distance $L = (Nv_0)^{1/3}$. Let (x_s, x_A) , $A = 1, \dots, l-1$, be a chart of \mathcal{Z}_{v_0} . We have chosen one coordinate to be the entropy per particle, while the remaining ones can be chosen freely. For later symbolic and interpretative convenience we impose the variables x_A to be dimensionless. Now, given an arbitrary point p in \mathcal{Z} with a given value of the coordinate v , we can imagine to perform an expansion (or contraction) of the volume in a time-scale τ_{fr} such that at the end of the process $v = v_0$, so the system will occupy a point of \mathcal{Z}_{v_0} , with coordinates (x_s, x_A) . Thus we can define the map from \mathcal{Z} to \mathbb{R}^{l+1}

$$p \mapsto (v, x_s, x_A). \quad (\text{A1})$$

Note that x_s coincides with the entropy per particle of p , because it is conserved along the transformation. Since in the transformation no entropy production occurs, the process is reversible, which means that the opposite transformation $v_0 \rightarrow v$ cannot be distinguished from the time-reversed of the transformation $v \rightarrow v_0$. This implies that if we make the cycle of expansion and contraction $v \rightarrow v_0 \rightarrow v$ in a time-scale τ_{fr} we should end in the initial state. Thus two distinct points p and p' with the same v have to be sent, in the transformation $v \rightarrow v_0$, into two distinct points of \mathcal{Z}_{v_0} , implying that the map (A1) is one-to-one and represents a global chart of \mathcal{Z} .

We have defined a convenient coordinate system in which the curves explored making fast expansions are given by the conditions $x_s = \text{const}$ and $x_A = \text{const}$. To see this it is sufficient to note that two points p and p' belonging to the same curve, with different volumes v and v' , end up in the same point of \mathcal{Z}_{v_0} when their respective volumes are sent into v_0 . Thus they have the same value of x_s and x_A .

The generator of the curves $x_s, x_A = \text{const}$ is the vector field

$$W_{\text{fr}} := \left. \frac{\partial}{\partial v} \right|_{x_s, x_A}, \quad (\text{A2})$$

which, applied to any thermodynamic function, describes its variation in an expansion occurring in a time-scale τ_{fr} and therefore must coincide with W_{fr} introduced in subsection 2.2.

A2. *The invertibility of the relaxation-time matrix*

We will prove the invertibility of $\delta^A_B + i\omega\tau^A_B$ with a reduction ad absurdum. Let us assume that there is a non-zero complex vector p^B such that

$$(\delta^A_B + i\omega\tau^A_B)p^B = 0. \tag{A3}$$

Then we have that

$$\tau^A_B\tau^B_C p^C = -\omega^{-2}p^A. \tag{A4}$$

Now consider the differential (58). Since it describes an exact one-form, the equivalence between the second mixed derivatives produces the Maxwell relation

$$\left. \frac{\partial x_A}{\partial \mathbb{A}^B} \right|_{v,x_s} = \left. \frac{\partial x_B}{\partial \mathbb{A}^A} \right|_{v,x_s}. \tag{A5}$$

Using (73) and (75), this implies

$$\Xi_{AC}\tau^C_B = \tau^C_A\Xi_{CB}. \tag{A6}$$

Contracting with $(p^A)^*\tau^B_D p^D$ we obtain

$$\Xi_{AC}(p^A)^*(\tau^C_B\tau^B_D p^D) = (p^A)^*\tau^C_A\Xi_{CB}\tau^B_D p^D, \tag{A7}$$

which, using (A3) and (A4), becomes

$$-\Xi_{AC}(p^A)^* p^C = \Xi_{CB}(p^C)^* p^B. \tag{A8}$$

Considering that Ξ_{AB} is a symmetric positive definite matrix and p^A is different from zero we have a contradiction.

A3. *Bulk viscosity of non(ultra)-relativistic ideal gases*

We have constructed the coordinates x_A in a way that $W_{fr}(x_A) = 0$, see equation (9). Expressing this condition in the chart (v, x_s, \mathbb{A}^B) we find

$$\left. \frac{\partial x_A}{\partial v} \right|_{\mathbb{A}^B, x_s} + \sum_{B=1}^{l-1} \left. \frac{\partial x_A}{\partial \mathbb{A}^B} \right|_{v, x_s} W_{fr}(\mathbb{A}^B) = 0. \tag{A9}$$

The requirement that the curves generated by W_{fr} starting in the equilibrium surface (given by the conditions $\mathbb{A}^B = 0$) are entirely contained in the equilibrium surface itself is expressed by the tangentiality condition $W_{fr}(\mathbb{A}^B) = 0 \forall B$ whenever all the \mathbb{A}^B vanish, leading to

$$\left. \frac{\partial x_A}{\partial v} \right|_{\mathbb{A}^B=0, x_s} = 0. \tag{A10}$$

As it can be seen from equation (65), the condition (A10) immediately implies that Π vanishes. Furthermore, one can easily verify from equation (71) that if a fluid element is prepared in a local equilibrium state, then $\mathbb{A}^B = 0 \forall B$ along its worldline, provided that (A10) holds. Thus we have proven that the bulk viscosity is zero whenever $W_{fr}(\mathbb{A}^A) = 0$ in equilibrium.

However this must always be true for non-relativistic and ultra-relativistic gases. In fact, starting from (10), one can immediately prove the Maxwell relation

$$\left. \frac{\partial \mathbb{A}^A}{\partial v} \right|_{x_s, x_B} = \left. \frac{\partial \Psi}{\partial x_A} \right|_{v, x_s}. \tag{A11}$$

However for a non(u)lra-relativistic gas we have that, independently from the fact that the fluid element is in thermodynamic equilibrium or not,

$$\Psi = (\Gamma - 1)(\mathcal{U} - mn). \tag{A12}$$

This is a kinetic identity which holds for any isotropic f . Note that in the ultra-relativistic case one has to formally set $m = 0$. Using (A12), equation (A11) becomes

$$W_{fr}(\mathbb{A}^A) = -(\Gamma - 1)n\mathbb{A}^A, \tag{A13}$$

which implies that in equilibrium

$$W_{fr}(\mathbb{A}^A) = 0. \tag{A14}$$

A4. Affinity of a diluted relativistic gas

According to (10) the affinity associated to ζ_c is

$$\mathbb{A} = - \left. \frac{\partial \tilde{\mathcal{U}}}{\partial \zeta_c} \right|_{v, x_s} = - \left. \frac{\partial \tilde{\mathcal{U}}}{\partial \zeta_c} \right|_{\lambda} - \left. \frac{\partial \tilde{\mathcal{U}}}{\partial \lambda} \right|_{\zeta_c} \left. \frac{\partial \lambda}{\partial \zeta_c} \right|_{v, x_s}. \tag{A15}$$

Using the results of section 8, it is possible to check that the affinity can be written as the sum of three terms:

$$\mathbb{A} = \mathbb{A}_{(1)} + \mathbb{A}_{(2)} + \mathbb{A}_{(3)}, \tag{A16}$$

with

$$\begin{aligned} \mathbb{A}_{(1)} &= \tilde{\mathcal{U}}\phi' \\ \mathbb{A}_{(2)} &= m e^{-\phi} \int_0^{+\infty} \xi^2 e^{-\zeta_c \sqrt{1+\xi^2}} \sqrt{1+\xi^2} \sqrt{1+\frac{\xi^2}{\lambda^2}} d\xi \\ \mathbb{A}_{(3)} &= 3\Theta \left. \frac{\partial}{\partial \zeta_c} (\ln \lambda) \right|_{v, x_s} = -\Theta \zeta_c \phi''. \end{aligned} \tag{A17}$$

Now we only need to check that for $\lambda = 1$ we find $\mathbb{A} = 0$. Note that for $\lambda = 1$

$$\tilde{\mathcal{U}} = -m\phi' \quad \Theta = \frac{m}{\zeta_c} \tag{A18}$$

and

$$\mathbb{A}_{(2)} = m(\phi'' + \phi'^2). \tag{A19}$$

When we plug these formulas into the expression for \mathbb{A} we find that the three terms cancel out, leaving

$$\mathbb{A} = 0, \tag{A20}$$

which is what we wanted to prove.

A5. The second derivative of the energy in the generalised reaction coordinate

Using (A16) and the definition of \mathbb{A} we see that we need to compute

$$\left. \frac{\partial^2 \tilde{\mathcal{U}}}{\partial \zeta_c^2} \right|_{v, x_s} = - \left. \frac{\partial \mathbb{A}_{(1)}}{\partial \zeta_c} \right|_{v, x_s} - \left. \frac{\partial \mathbb{A}_{(2)}}{\partial \zeta_c} \right|_{v, x_s} - \left. \frac{\partial \mathbb{A}_{(3)}}{\partial \zeta_c} \right|_{v, x_s}. \quad (\text{A21})$$

However we need to compute the result only in equilibrium, so we can use this fact to simplify the formulas, for example

$$\left. \frac{\partial \mathbb{A}_{(1)}}{\partial \zeta_c} \right|_{v, x_s} = -\mathbb{A} \phi' + \tilde{\mathcal{U}} \phi'' = -m \phi' \phi''. \quad (\text{A22})$$

We have employed the fact that in equilibrium the affinity vanishes and that the first equation of (A18) holds.

With direct calculations one can show that in equilibrium

$$\left. \frac{\partial \mathbb{A}_{(2)}}{\partial \zeta_c} \right|_{v, x_s} = m \phi''' + 2m \phi' \phi'' + \frac{m \zeta_c \phi''}{3} (\phi'' + \phi'^2 - 1) \quad (\text{A23})$$

and

$$\left. \frac{\partial \mathbb{A}_{(3)}}{\partial \zeta_c} \right|_{v, x_s} = -m \phi''' + m \phi' \phi'' - \frac{m \phi''}{\zeta_c} + \frac{m \zeta_c \phi''}{3} (\phi'^2 - 1) - \frac{m J}{3} (\zeta_c \phi'')^2. \quad (\text{A24})$$

Plugging these results into (A21) we find

$$\left. \frac{\partial^2 \tilde{\mathcal{U}}}{\partial \zeta_c^2} \right|_{v, x_s} = m \phi'' \left[\frac{1}{\zeta_c} - 2\phi' - \frac{\zeta_c}{3} (\phi'' + 2\phi'^2 - 2) + \frac{J \zeta_c^2 \phi''}{3} \right]. \quad (\text{A25})$$

Using the formulas of (185) it can be further simplified to reach the form

$$\left. \frac{\partial^2 \tilde{\mathcal{U}}}{\partial \zeta_c^2} \right|_{v, x_s} = \frac{m \zeta_c \phi''^2}{3} \left(1 + \zeta_c J - \frac{3}{\zeta_c^2 \phi''} \right). \quad (\text{A26})$$

A6. The moments of single-particle distribution

Working in the reference of the matter element we have that f can be seen as a function of p^0 . In particular we can introduce

$$I := \frac{dn}{dp^0} = 4\pi |\mathbf{p}| p^0 f, \quad (\text{A27})$$

the density of particle per unit energy. Clearly it contains the same amount of information of f and it is easy to verify that

$$n = \int_0^{+\infty} I dp^0 \quad n \tilde{\mathcal{U}}_N = \int_0^{+\infty} (p^0)^N I dp^0. \quad (\text{A28})$$

Now let us study the series

$$F(z) := \sum_{N=0}^{+\infty} \frac{(-z)^N}{N!} n \tilde{\mathcal{U}}_N, \quad (\text{A29})$$

where z is an arbitrary complex number with positive real part. Under this condition we can bring the series in the integrals presented in (A28), giving

$$F(z) = \int_0^{+\infty} e^{-zp^0} I(p^0) dp^0. \quad (\text{A30})$$

This is the Laplace transform of I , which can be inverted, so we arrive at the formula

$$I = \mathfrak{L}^{-1}\{F\}(p^0), \quad (\text{A31})$$

where \mathfrak{L}^{-1} is the inverse Laplace transform. This proves that, if n and all the $\tilde{\mathcal{U}}_N$ are known, then it is possible to reconstruct I and therefore f .

Appendix B. Absence of a macroscopic criterion to constrain the temperature

In this appendix we explain the physical origin of the ambiguity in the definition of the temperature exposed in subsection 2.5.

Let us consider the homogeneous fluid in the box described in subsections 2.1 and 2.2. If we take the limit

$$\tau_M \longrightarrow +\infty, \quad (\text{B1})$$

the variables x_A can be considered constants of motion. We are essentially switching off the microscopic processes at the origin of their relaxation. Imagine in this limit to put the system in contact with an ideal constant temperature heat bath, with temperature $\Theta_H = \text{const}$. This is an effectively infinite mass–energy reservoir characterized by the equation of state

$$E_H(S_H) = E_{H0} + \Theta_H S_H, \quad (\text{B2})$$

where E_H is the energy measured in the frame of the walls. The second principle of thermodynamics, combined with the conservation of the total energy, tells us that after an equilibration process the substance in the box will reach the state which minimizes the quantity

$$F = E - \Theta_H S. \quad (\text{B3})$$

Therefore, considering that N is assumed fixed (no exchange of particles happens), we find that we have to impose

$$\delta \tilde{\mathcal{F}} = \frac{\delta F}{N} = (\Theta - \Theta_H) \delta x_s + \mathcal{K} \delta v - \sum_{A=1}^{l-1} \mathbb{A}^A \delta x_A = 0. \quad (\text{B4})$$

Imposing that the walls are fixed ($\delta v = 0$) and that the interaction with the bath does not destroy the conservation of x_A ($\delta x_A = 0$), the only condition we get is

$$\Theta = \Theta_H. \quad (\text{B5})$$

When a system is in thermal equilibrium with a bath, its temperature must coincide with Θ_H , so we can interpret Θ as the generalization of the notion of temperature to quasi-equilibrium states.

Now let study the evolution of the variables y_B , introduced in (18), during the process of equilibration with the bath. Considering that v and x_A are constant, we find

$$\frac{dy_B}{dt} = \frac{\partial y_B}{\partial v} \frac{dv}{dt} + \frac{\partial y_B}{\partial x_s} \frac{dx_s}{dt} + \sum_{A=1}^{l-1} \frac{\partial y_B}{\partial x_A} \frac{dx_A}{dt} = \frac{\partial y_B}{\partial x_s} \frac{dx_s}{dt}. \tag{B6}$$

Now let us assume that the y_B are built in a way that $\Theta \neq \Theta'$, see (20). Then there must be a B such that

$$\frac{\partial y_B}{\partial x_s} \neq 0, \tag{B7}$$

so, in general, during the evolution we will have

$$\frac{dy_B}{dt} \neq 0. \tag{B8}$$

This reveals where the ambiguity in the definition of the temperature comes from: since the interaction with the bath modifies the values of energy and entropy of the fluid, if we assume that the x_A are constant we will necessarily find that some of the y_B are no more conserved and vice versa. Therefore the ambiguity in the definition of the temperature reflects the fact that there is no macroscopic criterion to decide which set of possible variables x_A is conserved in the interaction with an ideal heat bath. This kind of problem has implications also in the definition of the equilibrium temperature in a relativistic context as it is the fundamental origin of the famous Planck–Ott imbroglio [64].

To solve the ambiguity, if possible, one has to study the dissipative processes involved in the relaxation of the system to thermodynamic equilibrium and understand if there is a particular choice of the variables x_A which is more reasonable to be conserved in an interaction with an ideal heat bath. This is a problem of kinetic theory which goes beyond a pure thermodynamic description.

Let us assume that an unambiguous criterion to determine whether a state variable is conserved in the interaction with the bath exists and that there are two alternative complete sets of $l - 1$ coordinates, x_A and y_B (conserved in the adiabatic expansions), satisfying this criterion. Then combining the constraint (21) with the fact that the left-hand side of (B6) must be equal to zero, we find that

$$y_B = y_B(x_A), \tag{B9}$$

which implies

$$\mathcal{K} = \mathcal{K}' \quad \Theta = \Theta', \tag{B10}$$

removing all the ambiguities.

Note that, even in this case, the chemical potential is not gauge invariant out of equilibrium. In fact, from (15), it is easy to see that, if (B10) holds, then

$$\mu - \sum_{A=1}^{l-1} \mathbb{A}^A x_A = \mu' - \sum_{B=1}^{l-1} \mathbb{B}^B y_B. \tag{B11}$$

This can be independently derived from the observation that we can define the quantity

$$\mu_{\text{tot}} := \left. \frac{\partial \mathcal{U}}{\partial n} \right|_{s,x_A} = \left. \frac{\partial \mathcal{U}}{\partial n} \right|_{s,y_B}, \tag{B12}$$

which can be considered a gauge-invariant (for chemical gauges of the kind (B9)) chemical potential of the fluid.

Appendix C. Adiabatic expansion of ideal gases

In this appendix we derive the formula for the transformation of the particle distribution function f under an adiabatic expansion. Our derivation consists of two steps.

Consider a single particle which is bouncing inside a cubic box of volume $V = L^3$. Suppose that the box is expanding infinitely slowly, then we have that

$$\frac{dp_j}{p_j} = -\frac{dL}{L}, \tag{C1}$$

where p_j is the absolute value of the component j of the momentum of the particle (the component of the momentum changes sign in any collision with the walls, therefore we consider only the absolute value). We give two proofs of this formula.

Proof from special relativity

Let us study what happens when the particle collides with the moving wall which is normal to the direction 1. Before the collision the four-momentum of the particle is $(\epsilon, p^1, p^2, p^3)$. It is convenient to boost to the wall's frame, which is moving with infinitesimal velocity w . In this frame the four-momentum of the particle is

$$(\gamma\epsilon - \gamma w p^1, \gamma p^1 - \gamma w \epsilon, p^2, p^3), \tag{C2}$$

with $\gamma = (1 - w^2)^{-1/2}$. The walls are perfectly reflecting, therefore in this frame after the collision we have

$$(\gamma\epsilon - \gamma w p^1, -\gamma p^1 + \gamma w \epsilon, p^2, p^3). \tag{C3}$$

Transforming back to the original frame we find that the momentum after the collision is

$$p_1^{ac} = \gamma^2(-p_1 + 2w\epsilon - w^2 p_1) \quad p_2^{ac} = p_2 \quad p_3^{ac} = p_3. \tag{C4}$$

Considering that w is infinitesimal we can neglect the order w^2 , obtaining that

$$p_1^{ac} = -(p_1 - 2w\epsilon), \tag{C5}$$

therefore the variation of the modulus of the momentum induced by a collision is

$$(dp)_{\text{coll}} = -2w\epsilon. \tag{C6}$$

Assuming that only one of the two opposite walls is moving, then in a time dt (sufficiently long to have many collisions with the wall, but sufficiently short to produce a small displacement of the wall) we have that

$$dL = w dt. \tag{C7}$$

The number of collisions of the particle with the wall is

$$N_{\text{coll}} = \frac{v^1 dt}{2L} = \frac{v^1 dL}{2wL}, \quad (\text{C8})$$

where v^1 is the absolute value of the component 1 of the velocity of the particle. The variation of the modulus of the momentum during dt is

$$dp_1 = N_{\text{coll}}(dp)_{1\text{coll}} = -\frac{\epsilon v^1 dL}{L}. \quad (\text{C9})$$

In the context of special relativity it must be true that $p_1 = \epsilon v^1$ both in the massive and massless case, so we find equation (C1).

Proof from quantum mechanics

The Hamilton operator of the particle in the box has the form

$$\hat{H} = \epsilon(\hat{\mathbf{p}}) + V_w(\hat{\mathbf{x}}). \quad (\text{C10})$$

ϵ is the kinetic energy and depends only on the modulus of the momentum, V_w is the potential energy of the walls, it is zero inside the box and infinite outside. It is known that the eigenstates of such a Hamiltonian are the normal modes

$$\psi_{\mathbf{a}}(\mathbf{x}) = \prod_{j=1}^3 \sqrt{\frac{2}{L}} \sin\left(\frac{\pi a_j}{L} x^j\right) \quad (\text{C11})$$

where $a_j \in \mathbb{N}$. The walls are located in $x^j = 0$ and $x^j = L$. The quantities

$$p_j = \frac{\hbar \pi a_j}{L}, \quad (\text{C12})$$

can be interpreted as the average moduli of the corresponding components of the momenta and it is true that

$$\hat{H}\psi_{\mathbf{a}} = \epsilon(\mathbf{p})\psi_{\mathbf{a}}. \quad (\text{C13})$$

Now, according to the adiabatic theorem, if the walls move infinitely slowly, then a particle which at the beginning of the expansion is occupying the eigenstate $\psi_{\mathbf{a}}$ at the end of the evolution will occupy the eigenstate $\psi'_{\mathbf{a}}$ associated to the final Hamiltonian. This means that if $L \rightarrow L'$, then

$$p_j = \frac{\hbar \pi a_j}{L} \rightarrow p'_j = \frac{\hbar \pi a_j}{L'}, \quad (\text{C14})$$

or, alternatively,

$$d(p_j L) = 0, \quad (\text{C15})$$

which is equivalent to (C1).

Now that we have proven how the momentum of a single particle in a box changes under an adiabatic expansion we can study the behaviour of f . Since we are supposing that during the process the two-body collisions do not have time to occur, the particles do not influence each

other, therefore they evolve as they were alone in the box. This means that if in the expansion $L \rightarrow \lambda L$, then, integrating (C1), the momentum of each particle transforms as follows:

$$\mathbf{p} \rightarrow \Phi_\lambda(\mathbf{p}) = \frac{\mathbf{p}}{\lambda}. \tag{C16}$$

The last step consists of finding the transformation law of f . Let us consider an arbitrary volume Ω in the momentum space. Before the expansion there are in the box

$$N(\Omega) = L^3 \int_\Omega f(\mathbf{p}) d_3p \tag{C17}$$

particles in this volume. After the transformation there is the same amount of particles in the volume $\Phi_\lambda(\Omega)$, therefore

$$L^3 \int_\Omega f(\mathbf{p}) d_3p = \lambda^3 L^3 \int_{\Phi_\lambda(\Omega)} f_\lambda(\mathbf{q}) d_3q, \tag{C18}$$

where f_λ is the distribution after the expansion and we have used the fact that the final volume is $\lambda^3 L^3$. Now, making change of variables and using the fact that the Jacobian determinant of Φ_λ is

$$J_{\Phi_\lambda} = \frac{1}{\lambda^3}, \tag{C19}$$

we find

$$\int_\Omega f(\mathbf{p}) d_3p = \int_\Omega f_\lambda\left(\frac{\mathbf{p}}{\lambda}\right) d_3p. \tag{C20}$$

Since this equality holds for any Ω we finally obtain our formula for the transformation of f under adiabatic expansion:

$$f_\lambda(\mathbf{p}) = f(\lambda\mathbf{p}). \tag{C21}$$

We conclude the appendix making a couple of fundamental coherence tests. We have shown that (38) always holds, therefore we need to check if it is satisfied by our formula for the adiabatic transformation. In the following we use a subscript λ to denote the quantities obtained from the distribution f_λ , and we do not put any subscript for the quantities computed with f .

With a change of coordinate $\mathbf{q} = \lambda\mathbf{p}$ you can easily see from (148) that

$$n_\lambda = \frac{n}{\lambda^3}, \tag{C22}$$

from (149) that

$$\mathcal{U}_\lambda = \frac{1}{\lambda^3} \int \epsilon\left(\frac{\mathbf{q}}{\lambda}\right) f(\mathbf{q}) d_3q \tag{C23}$$

and from (150) that

$$\Psi_\lambda = \frac{1}{3\lambda^4} \int q_j v^j \left(\frac{\mathbf{q}}{\lambda}\right) f(\mathbf{q}) d_3q \tag{C24}$$

The ratio between (C23) and (C22) gives us

$$\tilde{\mathcal{U}}_\lambda = v \int \epsilon\left(\frac{\mathbf{q}}{\lambda}\right) f(\mathbf{q}) d_3q. \tag{C25}$$

Taking the derivative with respect to λ and remembering (151) we find

$$\frac{d\tilde{U}_\lambda}{d\lambda} = -3v\lambda^2\Psi_\lambda, \quad (\text{C26})$$

which, considering that $v_\lambda = v\lambda^3$, is equivalent to (38).

The final coherence test consists of verifying that the entropy per particle is conserved in this transformation, proving that it is adiabatic. The first step of the proof consist of realising that the transformation (C21) implies that, see (153),

$$\sigma_\lambda(\mathbf{p}) = \sigma(\lambda\mathbf{p}). \quad (\text{C27})$$

Plugging this result in the formula for s_λ , see (152), and changing variable in the usual way we get

$$s_\lambda = \frac{s}{\lambda^3}. \quad (\text{C28})$$

Taking the ratio with (C22) we find that x_s does not vary with λ , which is what we wanted to prove.

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Coauthor Statement

Article under discussion:

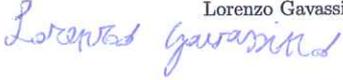
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Bulk viscosity in relativistic fluids: from thermodynamics to hydrodynamics

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Declaration of Lorenzo Gavassino (first author):

I, Lorenzo Gavassino, hereby declare that my contribution to the aforementioned research article amounts to 80%. I decided how to address the problem. I performed the mathematical analysis entirely. I also performed the numerical analysis, and created the plots. I wrote the draft and addressed the questions and comments of the referees.

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Declaration of Marco Antonelli (second author):

I, Marco Antonelli, hereby declare that my contribution to the aforementioned research article amounts to 10%. I introduced Lorenzo Gavassino (the first author) to Mathematica, which allowed him to perform the numerical analysis. I also contributed to the preparation of the draft, and suggested ways to improve the exposition.

Marco Antonelli


Declaration of Brynmor Haskell (third author):

I, Brynmor Haskell, hereby declare that my contribution to the aforementioned research article amounts to 10%. I introduced Lorenzo Gavassino (the first author) to the problem, providing the most important references, and fixing the final goal of the work. I also contributed to the preparation of the draft, and suggested some useful applications.

Brynmor Haskell



Chapter 6

**When the entropy has no maximum:
A new perspective on the instability of
the first-order theories of dissipation**

When the entropy has no maximum: A new perspective on the instability of the first-order theories of dissipation

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The first-order relativistic fluid theories of dissipation proposed by Eckart and Landau-Lifshitz have been proved to be unstable. They admit solutions which start in proximity of equilibrium and depart exponentially from it. We show that this behavior is due to the fact that the total entropy of these fluids, restricted to the dynamically accessible states, has no upper bound. As a result, these systems have the tendency to constantly change according to the second law of thermodynamics and the unstable modes represent the directions of growth of the entropy in state space. We, then, verify that the conditions of stability of Israel and Stewart's theory are exactly the requirements for the entropy to have an absolute maximum. Hence, we explain how the instability of the first-order theories is a direct consequence of the truncation of the entropy current at the first order, which turns the maximum into a saddle point of the total entropy. Finally, we show that recently proposed first-order stable theories, constructed using more general frames, do not solve the instability problem by providing a maximum for the entropy, but, rather, are made stable by allowing for small violations of the second law.

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I. INTRODUCTION

In the gravitational wave era [1] it is necessary, more than ever, to have at our disposal relativistic hydrodynamic theories of dissipation which are well-suited for numerical implementation. Heat conduction and viscosity, in particular bulk viscosity [2], are thought to play a major role in neutron star mergers, and understanding these transport processes is necessary for a reliable interpretation of the data [3,4]. Special relativistic fluid dynamics is also an essential tool in high-energy nuclear physics, where it is successfully used to describe the quark-gluon plasma formed in heavy ion collisions [5].

The literature is rich with alternative theories of dissipation [6–10], possessing different mathematical properties and physical insight, whose central purpose is mostly to solve the pathological aspects of the minimal models of [11,12]. However, born as the straightforward relativistic generalizations of Navier-Stokes and of the Fourier law [13], the theories of Eckart and Landau-Lifshitz still appear, intuitively, as a natural way of embedding viscosity and heat conduction in a relativistic framework.

The aim of this paper is to provide a physical perspective on the fundamental origin of incompatibility of this kind of straightforward Navier-Stokes approach with relativistic thermodynamics. Furthermore, understanding the cause of the incompatibility will allow us to interpret the physical content of the most important modern theories of dissipation depending on how they solve this structural inconsistency.

Among all the troublesome properties that a Navier-Stokes-type theory exhibits (which include acausality and differential equations of a mixed hyperbolic-parabolic form), we will focus here on the instability of the equilibrium. Hiscock and Lindblom [14] have shown that, if homogeneous perfect-fluid configurations are slightly perturbed, the disturbance can grow with no bound, producing runaway solutions. This phenomenon is in contrast with our understanding of dissipation as the process which leads thermodynamic systems to converge to the equilibrium state as time goes to infinity.

The physical interpretation of this instability has never been completely clarified. Some qualitative studies were carried out by Carter [15] for the case of the heat conduction, which led to the conclusion that these anomalous behaviors might arise from an improper redistribution of the inertia between the particle and the entropy current. However, the apparently natural “regular” solution that these arguments seemed to suggest [16] has been proven to lead to a fluid model which is in turn unstable [17].

The most well known successful alternative to the Eckart and Landau-Lifshitz theories is the Israel-Stewart second-order theory [6], which has been shown to be causal and stable (for linear perturbations from equilibrium), if appropriate choices of the parameters are adopted [18]. Interestingly, [9,19,20] recently proved that, if different hydrodynamic frames from those considered by [11,12] are considered, stability and causality may be actually restored in a first-order theory. This unexpected result reveals that

the pathological behavior of the models of Eckart and Landau-Lifshitz does not arise directly from the first-order truncation, but must have a more subtle origin.

Since both the second-order theories and the general-frame first-order theories admit the original formulations of Eckart and Landau-Lifshitz as particular cases, they can be stable only within a particular range of values of their parameters. In both cases the stability conditions can be obtained only through a detailed perturbative analysis about equilibrium [18,21] and the conditions one obtains lack an intuitive physical interpretation [22]. This has led some authors to consider the Israel-Stewart theory too complicated and artificial and to claim that the thermodynamic background is not sufficiently understood [23].

In this paper we clarify both the physical origin of the instability of the first-order theories and the thermodynamic meaning of the stability conditions of the models of Israel and Stewart [6] and Bemfica *et al.* [20].

Throughout the paper we adopt the spacetime signature $(-, +, +, +)$ and work in natural units $c = k_B = 1$.

II. MATHEMATICAL PRELIMINARIES

To understand the origin of the instability of Eckart and Landau-Lifshitz theories we first need to know, from a mathematical perspective, why this phenomenon is not expected to occur in real dissipative systems. In this section we briefly recall the foundations of irreversible thermodynamics and set the stage for our discussion.

A. Entropy as a Lyapunov function

Given a set of dynamical variables z^j , which obey first-order equations of motion of the form

$$\dot{z}^j = \mathcal{F}^j(z^k), \quad (1)$$

we say that a smooth (i.e., continuous, differentiable and with continuous partial derivatives) function

$$S = S(z^j) \quad (2)$$

is a Lyapunov function of the system if

- (i) S admits an absolute maximum, i.e., there is a state defined by the values z_{eq}^j such that

$$S(z^j) \leq S(z_{\text{eq}}^j) \quad \forall z^j; \quad (3)$$

- (ii) the maximum value of S is reached only in ($z^j = z_{\text{eq}}^j$), so the point of absolute maximum is unique;

- (iii) S is a nondecreasing function of time:

$$\frac{dS}{dt} = \frac{\partial S}{\partial z^j} \mathcal{F}^j \geq 0. \quad (4)$$

When the system admits a function S of this kind, then the state (z_{eq}^j) is an equilibrium state of the system. In fact, if the system has an initial condition

$$z^j(0) = z_{\text{eq}}^j, \quad (5)$$

then S has its maximum possible value at $t = 0$. Since S cannot decrease and (z_{eq}^j) is the only state in which S is maximum, then we necessarily have

$$z^j(t) = z_{\text{eq}}^j \quad \forall t \geq 0. \quad (6)$$

The state of equilibrium can also be shown to be Lyapunov stable [24], namely for any $\epsilon > 0$ there is a $\kappa > 0$ such that, if

$$\delta_{jk}(z^j(0) - z_{\text{eq}}^j)(z^k(0) - z_{\text{eq}}^k) \leq \kappa^2, \quad (7)$$

where δ_{jk} is the Kronecker delta symbol, then

$$\delta_{jk}(z^j(t) - z_{\text{eq}}^j)(z^k(t) - z_{\text{eq}}^k) \leq \epsilon^2 \quad (8)$$

for any $t \geq 0$. Intuitively, this means that if the system starts “close enough” to the equilibrium state, then it will remain “close enough” forever. This is due to the fact that, to run away from equilibrium, the system should make S decrease, which is forbidden. Therefore, a system which is Lyapunov stable does not admit runaway solutions from equilibrium, but only solutions which converge to it, or evolve around it moving on surfaces at constant S .

In the kinetic theory of ideal gases, for finite systems governed by Boltzmann’s transport equation, the entropy (defined as minus Boltzmann’s H-functional, see e.g., [25]) satisfies all the requirements to be a Lyapunov function [26] over the state-space with fixed constants of motion, namely the total energy, linear momentum, angular momentum and possibly particle numbers or (in the case of a gas of quasi-particles) “the superfluid velocity” [25,27]. For this reason, once the constants of motion are assigned in the initial conditions, there is a unique equilibrium state and this state is guaranteed to be Lyapunov stable.

In a generic thermodynamic system, the existence, uniqueness and stability of the global thermodynamic equilibrium are ensured only if (for fixed values of the constants of motion, which will appear in the equation of state of the system [28]) the entropy is a Lyapunov function, see [29,30]. For this to be true, all the defining conditions (i,ii,iii) need to be fulfilled. When hydrodynamic models are formulated, however, the validity of the

third requirement (which is nothing but the second law of thermodynamics) is usually enforced by construction, while in many cases the conditions (i) and (ii) are not.

The main goal of this paper is to show that the instability of the Eckart and Landau-Lifshitz first-order theories is the result of the fact that in these theories the entropy does not have a maximum value and it may diverge even if the total energy and momentum of the system are conserved. Therefore, the existence of runaway solutions has a clear thermodynamic origin and is rooted in the fact that it is favorable for the system to depart from the perfect fluid state because this leads to an increase of entropy.

B. The degrees of freedom of the theory

Since our aim is to study the entropy as a function over all the configurations that the system is allowed explore, it is necessary to analyze this configuration space in detail. Throughout the paper we will consider a fluid whose energy-momentum tensor can be decomposed into a perfect-fluid part and a nonequilibrium deviation $\mathfrak{T}^{\mu\nu}$ as follows

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu} + \mathfrak{T}^{\mu\nu}. \quad (9)$$

The fluid is supposed not to interact with any other external field, so that we can impose energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0. \quad (10)$$

With the only exception of subsections [VI A](#), [VI B](#) and [VI F](#), we will always assume for simplicity that the particle number is not conserved (zero chemical potential) and that the fluid in thermodynamic equilibrium behaves as an ideal gas of ultra-relativistic particles. Therefore, the equilibrium pressure P and the equilibrium internal energy ρ are related by the kinetic identity

$$P = \frac{1}{3}\rho \quad (11)$$

and a radiation-type equation of state

$$\rho = a_R \Theta^4 \quad (12)$$

holds, where Θ is the (equilibrium) temperature and a_R is a constant. The rest-frame (equilibrium) entropy density s can be obtained from the Euler relation

$$s\Theta = \rho + P, \quad (13)$$

which immediately implies

$$s = \frac{4}{3}a_R \Theta^3. \quad (14)$$

This choice of fluid is made just to have a reference model in which all the calculations can be easily performed analytically. In fact, our purpose is not to give another proof of the instability of the first-order theories (which is a well-known fact), but to understand its thermodynamic meaning. Our simplified model, therefore, will only serve as a guiding example to the mechanisms of the instability, but the most important results of the paper will be shown (when necessary) to hold in full generality.

The total flow of entropy is assumed to be described by an entropy four-current

$$s^\mu = s u^\mu + \sigma^\mu, \quad (15)$$

where σ^μ is a nonequilibrium contribution, which in general vanishes when $\mathfrak{T}^{\mu\nu} = 0$. The second law of thermodynamics has the local form [\[31,32\]](#)

$$\nabla_\mu s^\mu \geq 0. \quad (16)$$

In the following, we will work for simplicity in a flat spacetime with global inertial coordinates. Therefore, assuming that the fluid occupies a finite volume, we define the total entropy of the system at a given time as

$$S = \int s^0 d_3x. \quad (17)$$

Equation [\(16\)](#), then, implies

$$\frac{dS}{dt} \geq 0, \quad (18)$$

which is the second law in its global form.

In a general hydrodynamic model, the state of the fluid at a given time can be completely assigned by determining the values of all the hydrodynamic fields in that particular instant of time. Therefore, the state-space of the system is the set of all the possible fluid configurations. In our example, to specify the configuration of the fluid completely, we need to know at least four independent equilibrium quantities, such as Θ and the three spatial components u^j of the four-velocity (from now on we adopt the notation that the index j runs over the spatial components only: $j = 1, 2, 3$). Regarding the degrees of freedom introduced by the dissipative terms, one needs to make a more careful analysis.

Let us focus, for definiteness, on the case of bulk viscosity, given by a choice of the nonequilibrium contributions of the form

$$\mathfrak{T}^{\mu\nu} = \Pi(g^{\mu\nu} + u^\mu u^\nu). \quad (19)$$

The scalar Π is the viscous stress. In Newtonian hydrodynamics, the value of Π is usually determined from the Navier-Stokes assumption

$$\Pi = -\zeta\partial_j u^j. \quad (20)$$

This implies that, in Newtonian hydrodynamics, if we assign the value of u^j everywhere (on the hypersurface at constant time), then the value of Π is automatically determined (analogous arguments hold for the shear stress and the heat flux, assuming the Fourier law). Therefore, in Newtonian hydrodynamics, the presence of dissipation does not introduce new degrees of freedom.

In relativistic hydrodynamics, however, equation (20) cannot hold in every reference frame, because it is not covariant. The most trivial relativistic generalization of (20) is provided by the first-order prescription

$$\Pi = -\zeta\partial_\nu u^\nu. \quad (21)$$

The presence of a term $\partial_t u^0$ implies that the knowledge of the four-velocity along the hypersurface at constant time is not enough to constrain the value of Π . Instead, the foregoing equation can be inverted as follows,

$$\partial_t u^0 = -\partial_j u^j - \frac{\Pi}{\zeta}, \quad (22)$$

which shows that in relativity Π can be considered a new degree of freedom of the model [14] and (22) is the new equation of motion that closes the system.¹ Therefore, in a relativistic model for bulk viscosity, we need 5 independent hydrodynamic fields to completely specify the state (i.e., Θ , u^j and Π).

This relativistic enlargement of the state-space occurs whenever one constructs, in a preferred reference frame, a parabolic equation of the kind $(A\partial_t + B\partial_x^2)f = 0$ and then moves to a generic frame through a Lorentz boost. The derivatives in space become, in the new reference frame, linear combinations of derivatives in both space and time, producing a term proportional to $B\partial_t^2 f$ and therefore increasing the number of degrees of freedom of the model. More details about this mechanism can be found in Appendix, where a brief analysis for the case of the diffusion equation (whose instability in relativity is formally identical to the instability of Landau-Lifshitz, see [33]) is provided. In the appendix we also give an intuitive explanation of the connection underlined by [18] between stability and causality (although the problem of causality is not explicitly addressed in the present work).

In summary, the approach of extended irreversible thermodynamics [34,35] of treating the dissipative terms as degrees of freedom is unavoidable in relativity. In this

¹It can be rigorously proven that (21) produces a new degree of freedom by verifying that if we replace Π by $-\zeta\nabla_\nu u^\nu$ in the energy conservation relation $\partial_\mu T^{\mu 0} = 0$, the resulting equation has a second order term $\partial_t \partial_t u^0$ (which has no Newtonian analogue). This implies that we need to specify also $\partial_t u^0$ (or equivalently Π) in the initial conditions.

sense, there is no conceptual difference between the second order theories (where the dissipative terms are promoted to degrees of freedom explicitly) and the first order theories (where, for the case of Eckart and Landau-Lifshitz, this is hidden behind the fact that in the rest frame of the fluid element there are no derivatives in time).

We will show that this inevitable extension of the state space is the origin of the instability. In fact, it produces a new class of available thermodynamic states of the total fluid (which have no Newtonian analogue), and this generates novel paths in the state space in which the entropy can grow without any bound.

III. INSTABILITY OF THE ECKART THEORY OF HEAT CONDUCTION

The first example we examine is the model for heat conduction proposed by Eckart [11].

A. The instability mechanism

An energy-momentum tensor of the form

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu \quad (23)$$

is assumed. The four-vector q^μ is the heat flow and satisfies the geometrical constraint

$$u_\mu q^\mu = 0, \quad (24)$$

while the entropy four-current is postulated to be

$$s^\mu = s u^\mu + \frac{1}{\Theta} q^\mu. \quad (25)$$

Therefore, this theory is formulated in the general form presented in subsection II B with

$$\mathfrak{T}^{\mu\nu} = q^\mu u^\nu + u^\mu q^\nu \quad \sigma^\mu = \frac{1}{\Theta} q^\mu. \quad (26)$$

Let us consider a homogeneous portion of fluid. Then the Eqs. (10) and (16) acquire the simpler form (we recall that we work in a flat spacetime with global inertial coordinates)

$$\partial_t T^{0\nu} = 0 \quad \partial_t s^0 \geq 0. \quad (27)$$

The first equation implies that the energy and momentum densities, defined respectively as

$$\mathcal{E} := T^{00} \quad \mathcal{P}^j := T^{0j}, \quad (28)$$

are necessarily conserved during the evolution of homogeneous fluids. The second equation of (27) is the local version of (18) for homogeneous systems. Note that in homogeneous fluids the quantity per unit volume which needs to increase to ensure the validity of the second law is

not the rest-frame entropy density s , but the entropy density s^0 , measured in the frame in which the fluid is homogeneous. It is important to keep this difference in mind, because we will see that the Lorentz contraction of volumes plays a role in the instability.

Now our aim is to show that, for fixed values of \mathcal{E} and \mathcal{P}^j , the density s^0 can become arbitrarily large by varying the remaining unconstrained hydrodynamic variables. In this way it will be automatically proven that the entropy has no upper bound and thus cannot be a Lyapunov function for the system.

It is sufficient to work with $\mathcal{P}^j = 0$. Then, we can use the invariance under rotations of the fluid element to restrict ourselves to the case in which

$$u^0 = \gamma \quad u^1 = \gamma v \quad u^2 = u^3 = 0, \quad (29)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}. \quad (30)$$

From (29), one can easily show that the constraint (24) explicitly reads

$$q^0 = vq^1. \quad (31)$$

Let us briefly count the degrees of freedom. According to the discussion of subsection II B, to specify the state of the fluid completely we need to know, for the case of heat conduction, the value of 5 variables (v , Θ , q^1 , q^2 , q^3). However, as we said before, there are 4 constraints (\mathcal{E} , \mathcal{P}^1 , \mathcal{P}^2 , \mathcal{P}^3). This means that the system is in principle allowed to visit a 1D manifold of states (which we can parametrize e.g., with v) compatibly with four-momentum conservation. So we need to study how s^0 varies along this manifold.

First of all, we note that the conditions $\mathcal{P}^2 = \mathcal{P}^3 = 0$, combined with the third equation of (29), immediately imply

$$q^2 = q^3 = 0. \quad (32)$$

Now, recalling Eq. (23), we need to employ the two remaining constraints

$$\begin{aligned} \mathcal{E} &= 4P\gamma^2 - P + 2\gamma q^0 \\ \mathcal{P}^1 &= 4P\gamma^2 v + \gamma q^1 + \gamma v q^0 = 0 \end{aligned} \quad (33)$$

to write Θ and q^1 as functions of v . Note that we have used the kinetic identity (11) to substitute ρ with $3P$. With a little algebra, and with the aid of equation (31), one can show that the system (33) is equivalent to

$$P = \frac{1 + v^2}{3 - v^2} \mathcal{E} \quad q^1 = -\frac{4\gamma v}{3 - v^2} \mathcal{E}. \quad (34)$$

The second equation can be used to rewrite the zeroth component of (25) in the form

$$s^0 = \frac{\gamma s}{1 + v^2}. \quad (35)$$

Combining Eqs. (11), (12) and (14) one can show that

$$s = 4 \left(\frac{a_R \mathcal{P}^3}{3} \right)^{1/4}. \quad (36)$$

Using this expression into (35), together with the expression for the pressure given in (34), we finally obtain

$$s^0 = \tilde{s} (1 - v^2)^{-1/2} (1 + v^2)^{-1/4} \left(1 - \frac{v^2}{3} \right)^{-3/4}, \quad (37)$$

where we have defined

$$\tilde{s} := \frac{4}{3} (a_R \mathcal{E}^3)^{1/4}. \quad (38)$$

Now we immediately see that s^0 has no upper bound. In fact, as $v \rightarrow \pm 1$,

$$s^0 \rightarrow +\infty. \quad (39)$$

This proves that the entropy does not have a maximum, so it is not a Lyapunov function for Eckart's theory.

B. Physical interpretation of the instability

We can, now, provide a physical interpretation of the instability of Eckart's theory and why runaway solutions are admitted. Let us consider the homogeneous fluid configuration we presented in the previous subsection (with $\mathcal{P}^j = 0$) and let us impose $v = 0$. From (29) and (34) we find

$$u^j = 0 \quad P = \frac{1}{3} \mathcal{E} \quad q^j = 0. \quad (40)$$

These equations imply that the fluid is at rest ($u^j = 0$), in local thermodynamic equilibrium ($q^j = 0$), with rest-frame energy density $\rho = 3P = \mathcal{E}$. It is a well-known result of kinetic theory [26] that this fluid configuration should correspond to the state of global thermodynamic equilibrium, i.e., of absolute maximum entropy, compatibly with the conservation of the total energy and momentum. This configuration should therefore be stable, because any physically allowed (i.e., compatible with the conservation laws) spontaneous deviation from it would result in a decrease of the total entropy. In the case of Eckart's theory, however, this is not the case.

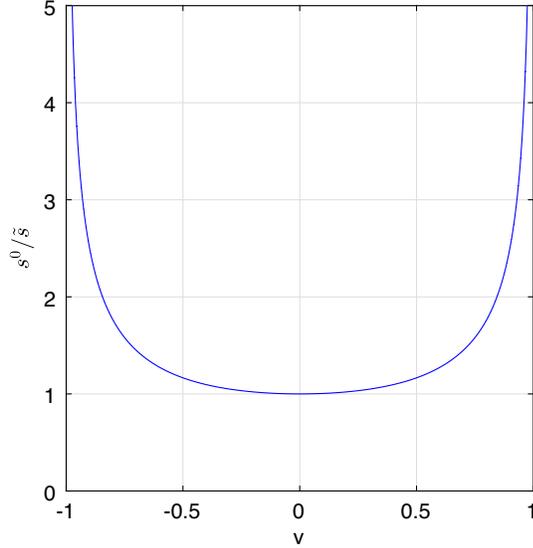


FIG. 1. Plot of the Eckart normalized entropy per unit volume s^0/\bar{s} as a function of the velocity v , as given in equation (37). As can be seen, the state $v = 0$ is not the maximum of the entropy density, but the minimum among the homogeneous configurations that the system is allowed to explore. As discussed in the main text, this is the origin of the instability.

To see this in more detail let us consider a state with a small $v > 0$. From the second equation of (34) we have

$$q^1 \approx -\frac{4}{3}\mathcal{E}v. \quad (41)$$

This corresponds to a configuration in which the fluid has accelerated in the positive direction 1, using the heat flux q^1 as rocket fuel. In fact, the total momentum is still zero because the energy flow of the heat in the negative direction 1 compensates the translational momentum of the fluid. We can compute the entropy in this state, expanding equation (37) to the second order, obtaining

$$s^0 \approx \bar{s} \left(1 + \frac{v^2}{2} \right). \quad (42)$$

We see that the entropy is an increasing function of v^2 . Now the origin of the instability is clear: since there is heat flux, entropy is necessarily produced, but if s^0 grows, then v^2 must increase, leading to a larger heat flux and therefore to a larger heat production. The fluid, then, accelerates more and more until it reaches the speed of light, where s^0 diverges, as can be seen in Fig. 1.

We have shown that the origin of the instability of Eckart's theory is fundamentally thermodynamic. Since the theory is constructed in a way to ensure the exact validity of the second law, the system will naturally evolve to the available state with maximum entropy. However, the perfect fluid state, identified by the condition $v = 0$, is

not the maximum of the entropy, but only a saddle point: restricting the entropy to the homogeneous configurations only, the perfect fluid state is the absolute minimum point.

C. The dynamics of the instability

The problems of Eckart's theory are often attributed to the Fourier-type law [36], namely the fact that the heat flow is expressed as [11]

$$q^\mu = -k\Theta(g^{\mu\nu} + u^\mu u^\nu) \left[\frac{\nabla_\nu \Theta}{\Theta} + u^\sigma \nabla_\sigma u_\nu \right], \quad (43)$$

where k is the conductivity coefficient. However, in our calculations we have never used explicitly the above expression. Since our analysis is purely thermodynamic (i.e., it deals only with the instantaneous properties of the system in a given state and not with its dynamical evolution in time), we see that the problem of Eckart's theory is more fundamental: it arises directly from the first-order expansion of the entropy current, Eq. (25). Equation (43) produces the instability only because it encodes the second law in the model.

Nevertheless, we now show that the mechanism for the instability we have presented in the previous subsection produces the homogeneous unstable mode of Eckart's theory identified by [14], proving the complete consistency of our analysis with the first-order stability studies.

Since we are working with global inertial coordinates, taking the component $\mu = 1$ of Eq. (43) gives

$$q^1 = -k(\partial_1 \Theta + u^1 u^\sigma \partial_\sigma \Theta + \Theta u^\sigma \partial_\sigma u^1). \quad (44)$$

We retain only the first order in v and use the fact that the configuration is homogeneous to find

$$q^1 = -k\Theta \dot{v}. \quad (45)$$

Combining with (41) we get the exponential growth law

$$\dot{v} = \Gamma_+ v, \quad (46)$$

with a rate

$$\Gamma_+ = \frac{4\mathcal{E}}{3k\Theta}. \quad (47)$$

This formula for Γ_+ is the formula of the imaginary frequency of the unstable mode identified by Hiscock and Lindblom [14], cf. Eq. (52) therein. Now, we can use the existence of this runaway solution to prove that the entropy density, for small v , needs to have the second-order expansion given in Eq. (42).

We consider the general equation for the entropy production of Eckart's theory,

$$\nabla_\nu s^\nu = \frac{q^\nu q_\nu}{k\Theta^2}. \quad (48)$$

At the second order in v it reduces to

$$\partial_t s^0 = \frac{q^1 q^1}{k\Theta^2}, \quad (49)$$

Using (45) and (46) we can recast it into the form

$$\partial_t s^0 = k\Gamma_+ v \dot{v}. \quad (50)$$

However it is easy to verify that

$$\tilde{s} = k\Gamma_+, \quad (51)$$

which, plugged into (50), gives

$$\partial_t s^0 = \frac{\tilde{s}}{2} \partial_t (v^2). \quad (52)$$

Then, considering that $s^0(v=0) = \tilde{s}$, we recover (42). Equation (52) shows that the instability (i.e., the fact that v^2 grows) is a direct result of the second law (i.e., the fact that s^0 grows) and this can only be true if the entropy is not maximal in $v=0$.

It is interesting to note that in this subsection we have followed an opposite path with respect to the previous ones. We have started directly from the first-order stability analysis of [14] and we have studied the runaway solution directly. Using Eq. (48), we have tracked how the entropy changes with time during the runaway, obtaining $s^0(t)$. Then, making the change of variable

$$s^0(t) = s^0(v(t)), \quad (53)$$

we obtained the function $s^0(v)$ directly. Hence, we have shown that the unstable modes probe the convexity of the entropy near $v=0$. In this sense, the existence of the runaway solutions is the marker of its saddle point nature, showing that, since the perfect fluid state is not Lyapunov stable, the entropy cannot be a Lyapunov function of the system. This argument is fully general and applies beyond the toy-model we are considering here.

D. Comparison with the Newtonian theory

The fact that the runaway solutions are a purely relativistic effect is now evident. In fact, the dependence of s^0 on v in the expansion (42) is an order v^2 (where we recall that v is measured in units of the speed of light). Furthermore, from (35) we also see that the convexity is due to the presence of coefficients like γ , which encodes the relativistic contraction of volumes, a phenomenon which does not exist in Newtonian physics. However, in the light of the discussion of subsection II B, it is now possible to

explain the fundamental mathematical difference between the two theories in more detail.

Let us consider the $0j$ component of the Eckart energy-momentum tensor (23):

$$\mathcal{P}^j = (\rho + P)u^0 u^j + q^0 u^j + q^j u^0. \quad (54)$$

To obtain the Newtonian limit we need to take the limit

$$\rho \rightarrow +\infty \quad u^0 \rightarrow 1, \quad (55)$$

obtaining the expression for the momentum density

$$\mathcal{P}^j = \rho u^j. \quad (56)$$

Hence, in the Newtonian limit

$$\mathcal{P}^j = 0 \Leftrightarrow u^j = 0. \quad (57)$$

Since in Newtonian physics the heat flux q^ν does not give any contribution to the momentum, this means that the fluid configurations with $v \neq 0$ cannot be explored when the total momentum density is zero. Therefore, the mode that gives rise to the instability [which is the way in which the system can probe the profile of (37)] is not dynamically allowed and the fluid is forced to remain at rest.

Again, this is just a reformulation of the statement that the state space of the relativistic fluid has a larger dimension with respect to the Newtonian one. A further confirmation comes from the fact that, as we see in (45), the unstable mode is made possible only because of the presence of the time derivatives in the relativistic Fourier law (43), which makes q^1 a degree of freedom of the relativistic theory.

IV. INSTABILITY OF FIRST-ORDER BULK VISCOSITY

The instability mechanism we have presented in the previous section is not a specific feature of Eckart's model for heat conduction, but is a general problem of both [11,12] theories. To see this, we perform an analogous study for the case of bulk viscosity. Note that both Eckart and Landau-Lifshitz theories treat this dissipative phenomenon in the same way, producing an instability which has been observed also in numerical simulations [37].

A. The instability mechanism

Let the stress-energy tensor be

$$T^{\mu\nu} = (\rho + P + \Pi)u^\mu u^\nu + (P + \Pi)g^{\mu\nu}, \quad (58)$$

and assume an entropy current

$$s^\mu = s u^\mu. \quad (59)$$

The non-equilibrium stress correction $\mathfrak{S}^{\mu\nu}$ is, then, given by (19), while the correction to the entropy current is $\sigma^\mu = 0$. The reference perfect fluid is again assumed to be the one introduced in subsection II B. We remark that the assumption that the fluid is an ultra-relativistic ideal gas must be considered in this section only a prescription for the equation of state and not a real microscopic interpretation, otherwise the bulk viscosity should vanish identically. Furthermore, again we focus on homogeneous configurations.

The line of reasoning is similar to that of subsection III A, with the difference that (as was shown by Hiscock and Lindblom [14]) the configuration in which the total momentum is zero is stable for homogeneous perturbations. Therefore, to see the unstable behavior of the fluid we need to work with an unperturbed state in which the fluid is moving. Without any loss of generality, we can impose \mathcal{E} , $\mathcal{P}^1 > 0$, and

$$\mathcal{P}^2 = \mathcal{P}^3 = 0. \quad (60)$$

The degrees of freedom of the system now are (v, Θ, Π) , while we have two relevant constraints $(\mathcal{E}, \mathcal{P}^1)$. Thus, again, we are dealing with a 1D manifold of physically accessible states. The constraint equations read

$$\begin{aligned} \mathcal{E} &= (4P + \Pi)\gamma^2 - P - \Pi \\ \mathcal{P}^1 &= (4P + \Pi)\gamma^2 v. \end{aligned} \quad (61)$$

These equations can be used to write P and Π as functions of v along the curve, giving

$$P = \frac{1}{3}(\mathcal{E} - \mathcal{P}^1 v) \quad \Pi = \frac{\mathcal{P}^1}{v} \left(1 + \frac{v^2}{3}\right) - \frac{4}{3}\mathcal{E}. \quad (62)$$

From the second equation we note that necessarily $v \neq 0$, which means that the system is dynamically allowed to exist only inside the open segment $0 < v < 1$. Setting $\Pi = 0$ in the second equation we obtain the speed at which the fluid moves in local thermodynamic equilibrium,

$$v_{\text{eq}} = \frac{2\mathcal{E} - \sqrt{4\mathcal{E}^2 - 3(\mathcal{P}^1)^2}}{\mathcal{P}^1}. \quad (63)$$

The first equation of (62) can be used to compute the entropy density s^0 as a function of v (performing analogous calculations to those presented in the previous section) giving

$$s^0 = \tilde{s}(1 - v^2)^{-1/2} \left(1 - \frac{\mathcal{P}^1}{\mathcal{E}} v\right)^{3/4}. \quad (64)$$

It is easy to show that v_{eq} is the only stationary point of s^0 . However, again, this point corresponds to the minimum of $s^0(v)$, as can be seen in Fig. 2.

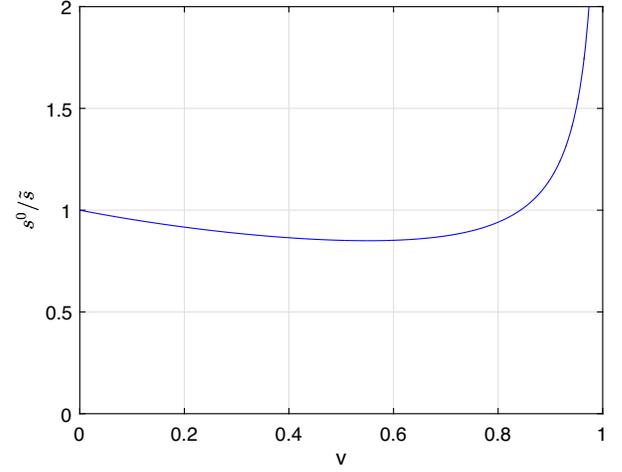


FIG. 2. The normalized entropy per unit volume s^0/\tilde{s} as a function of the velocity v , as given by Eq. (64), for $\mathcal{P}^1/\mathcal{E} = 2/3$. The state of local thermodynamic equilibrium is given by Eq. (63) (for $\mathcal{P}^1/\mathcal{E} = 2/3$ we find $v_{\text{eq}} = 3 - \sqrt{6} \approx 0.55$) and corresponds to the minimum of s^0/\tilde{s} . The particular value $\mathcal{P}^1/\mathcal{E} = 2/3$ has been selected for aesthetic reasons, the qualitative behavior of the function does not depend on this choice.

Now we are able to understand the instability of first-order bulk-viscous fluids. If the system starts with $v > v_{\text{eq}}$, then, since the entropy must grow, the fluid will be forced to accelerate until it reaches the speed of light. If, instead, it is prepared with $v < v_{\text{eq}}$, then it is forced to slow down until it stops. However, from the second equation of (62), we see that, as $v \rightarrow 0$, the viscous stress diverges, $\Pi \rightarrow \infty$, and, as Π explodes, so does the entropy production. Therefore this system, of equations approaches in a finite time a singularity.

It is clear why this instability mechanism is forbidden in Newtonian hydrodynamics: for these accelerations to be possible, the viscous stress needs to contribute to the total momentum, to ensure that the latter is constant during the evolution of the fluid.

V. HOW TO SOLVE THE PROBLEM

Since the theories of Israel and Stewart [6] and Bemfica *et al.* [20] have been proved to be (conditionally) stable, they manage to avoid the instability mechanisms we presented in the previous sections. To understand how this is possible, we need first of all to clarify what does not work in the formulation of the theories of Eckart [11] and Landau and Lifshitz [12] in the first place. In order to capture the physical essence of the problem, in this section we will deal with a thermodynamic toy-model which contains all the physical ingredients we need.

Consider a thermodynamic system with two degrees of freedom (u, q) , with no constants of motion and such that in a neighbourhood of the origin the entropy has the form (neglecting overall additive constants)

$$S = -u^2 + uq - q^2. \quad (65)$$

The function $S(u, q)$ is a quadratic form and can be rewritten in the diagonal representation

$$S = -\frac{1}{4}(u+q)^2 - \frac{3}{4}(u-q)^2. \quad (66)$$

Therefore, S admits a unique absolute maximum, given by

$$u = q = 0, \quad (67)$$

which is the stable equilibrium state of the system.

Now, let us assume that $|q| \ll |u|$ for every realistic initial condition of the system, so that we can expand S to the first order in q and neglect the contribution q^2 in (65). Hence, we obtain a first-order entropy

$$S_I = -u^2 + uq. \quad (68)$$

The eigenvalues of the Hessian of S_I are

$$\lambda_- = -1 - \sqrt{2} \quad \lambda_+ = -1 + \sqrt{2}. \quad (69)$$

Since $\lambda_+\lambda_- = -1 < 0$, the point $u = q = 0$ is no longer the maximum of the entropy, but it is a saddle point. Therefore, we see that neglecting second-order contributions to the entropy one may destroy its nature of Lyapunov function. This is what happens when a first-order theory is constructed: the omission of second-order terms in the entropy current can lead to the removal of contributions which are essential to determine the overall concavity of the total entropy, transforming its maximum into a saddle point.

There are two possible solutions to this problem. The first is to keep all the contributions to the second order. Clearly, if the microscopic input is realistic, all the terms should add up to give an entropy which is maximum in equilibrium (at least for small deviations from it), guaranteeing its Lyapunov stability. In fact, Sec. VI is devoted to show that the conditions of stability of the second-order theory of Israel and Stewart [6] are those which make the total entropy maximum in equilibrium. The same result holds also for the nonperturbative theories of Carter [8], Lopez-Monsalvo and Andersson [38], and Gavassino *et al.* [10], since they have the same stability properties as [6], see [39].

There is, however, an alternative approach. If one already knows that the equilibrium state must be $u = q = 0$, then a first-order model for dissipation should just ensure that the system naturally evolves to this state and that

$$\frac{dS_I}{dt} \geq 0 \quad (70)$$

only up to the first order in q . In other words, if S_I is just the first order expansion of the “real” S , then its growth should

be guaranteed except for terms of order q^2 . In this way one has the freedom to construct the equations of motion for u and q , tuning them in a way to ensure both the stability of the equilibrium, as an exact constraint, and the validity of the second law, as an approximate condition. Section VII is devoted to show that the possibility of small violations of the second law is the key to ensure the stability of the first-order theories in more general frames proposed by Bemfica *et al.* [19,20] and Kovtun [9].

We remark that in the present paper the terms *first-order* and *second-order theory* are interpreted according to the standard definition introduced by Hiscock and Lindblom [18] and currently used in textbooks [40]. The terminology refers to the order of the expansion of the entropy four-current used to derive the equations of motion (a procedure which always requires one to move to the second order, even in the construction of first-order theories, as shown by Hiscock and Lindblom [18]), not to the order in the displacements from equilibrium of the final hydrodynamic equations. Therefore, according to this definition, Israel-Stewart remains a second-order theory even when it is linearized for small deviations about equilibrium (the equations retaining the Cattaneo-type structure, see Appendix). As a consequence, the theories of heat conduction proposed by Andersson and Lopez-Monsalvo [41] and Ván and Biró [23], although referred to as first-order, are considered in the present paper as belonging to the class of second-order theories.

VI. STABILITY OF THE SECOND ORDER THEORIES

We show that the conditions of stability of the theory of Israel and Stewart [6] in the Eckart frame obtained by Hiscock and Lindblom [18] coincide with the condition for the total entropy of the fluid to be maximum in equilibrium (and thus to be a Lyapunov function).

A. Brief summary of the theory

In the second-order theory one assumes that the stress-energy tensor of the fluid has the form

$$T^{\mu\nu} = (\rho + P + \Pi)u^\mu u^\nu + (P + \Pi)g^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \Pi^{\mu\nu} \quad (71)$$

with

$$u_\mu q^\mu = u_\mu \Pi^{\mu\nu} = \Pi^{[\mu\nu]} = \Pi^\mu{}_\mu = 0. \quad (72)$$

The three dissipative contributions q^μ , Π , $\Pi^{\mu\nu}$ are respectively heat flux, bulk viscosity and shear viscosity. Furthermore, it is assumed that there is a conserved particle current $n^\mu = nu^\mu$ (we work in the Eckart frame) such that

$$\nabla_\mu n^\mu = 0. \quad (73)$$

The entropy current is expanded according to the logic of the extended irreversible thermodynamics approach [35]:

$$s^\mu = su^\mu + \frac{q^\mu}{\Theta} - \frac{1}{2}(\beta_0\Pi^2 + \beta_1q^\nu q_\nu + \beta_2\Pi^{\nu\rho}\Pi_{\nu\rho})\frac{u^\mu}{\Theta} + \alpha_0\Pi\frac{q^\mu}{\Theta} + \alpha_1\frac{\Pi^{\mu\nu}q_\nu}{\Theta}. \quad (74)$$

The quantity s is the equilibrium entropy, i.e., the entropy which is computed using the equilibrium equation of state

$$s = s(\rho, n) \quad (75)$$

of the fluid. This model is said to be a second-order theory because the entropy four-current is expanded to the second order in the deviations from local thermodynamic equilibrium. The second-order terms in q^μ , Π , $\Pi^{\mu\nu}$ appearing in equation (74) are, indeed, the novelty introduced in the model and their presence can compensate (if the coefficients satisfy appropriate conditions) the inevitable explosion of the zeroth and first order contributions we described in the previous sections.

The equations of motions, which we do not need to report here, are constructed in way to ensure that

$$\nabla_\mu s^\mu = \frac{q^\mu q_\mu}{k\Theta^2} + \frac{\Pi^2}{\zeta\Theta} + \frac{\Pi^{\mu\nu}\Pi_{\mu\nu}}{2\eta\Theta} \geq 0, \quad (76)$$

where k , ζ and η are respectively the heat conductivity, the bulk viscosity and the shear viscosity coefficient.

B. The stability conditions

Let us summarize the steps followed by Hiscock and Lindblom [18] to find the conditions for the second-order theory to admit a stable equilibrium.

First of all, they consider the stationary solutions to the system of equations, which are good candidates to be also equilibrium configurations. Clearly, for the configuration to be stationary, one needs to set the entropy production to zero. Thus from (76) they find that all the dissipative terms must vanish:

$$q^\mu = \Pi = \Pi^{\mu\nu} = 0. \quad (77)$$

Therefore, the energy-momentum tensor of the fluid in equilibrium must be that of a perfect fluid, in agreement with kinetic theory [26,31]. They also use the equations of motion explicitly to obtain the remaining equilibrium conditions which are provided by kinetic theory:

$$\nabla_\mu(u_\nu/\Theta) + \nabla_\nu(u_\mu/\Theta) = 0 \quad (78)$$

and

$$\nabla_\mu\left(\frac{\rho + P - \Theta s}{n\Theta}\right) = 0. \quad (79)$$

Once the candidates to be equilibrium states are found, they write the equations of motion for perturbations about these states. Subsequently, they show that it is possible to define a current E^μ (equation (44) in [18]) which is quadratic in the deviations and satisfies the condition

$$\nabla_\mu E^\mu = -\left[\frac{\delta q^\mu \delta q_\mu}{k\Theta^2} + \frac{(\delta\Pi)^2}{\zeta\Theta} + \frac{\delta\Pi^{\mu\nu} \delta\Pi_{\mu\nu}}{2\eta\Theta}\right] \leq 0, \quad (80)$$

where δf is the perturbation of a generic hydrodynamic variable f . Introducing, then, the functional

$$E = \int E^0 d_3x, \quad (81)$$

Eq. (80) implies

$$\frac{dE}{dt} \leq 0. \quad (82)$$

Finally, they find the conditions under which

$$E \geq 0 \quad (83)$$

for any small deviation from equilibrium, obtaining a set of constraints for the equation of state and the coefficients of the model. When these constraints are satisfied, then, combining (82) and (83), one obtains

$$E(t) \in [0, E(0)] \quad \forall t \geq 0, \quad (84)$$

implying, for arguments analogous to those we exposed by us in Sec. II A, the Lyapunov stability of the configuration.

C. Equivalence with the maximum entropy principle

At this point, showing that the stability conditions found by Hiscock and Lindblom [18] imply that the entropy is maximum at equilibrium is straightforward.

First of all, we note that (77) implies

$$\delta q^\mu = q^\mu \quad \delta\Pi = \Pi \quad \delta\Pi^{\mu\nu} = \Pi^{\mu\nu}. \quad (85)$$

Therefore we can combine (76) with (80) to obtain

$$\nabla_\mu s^\mu = -\nabla_\mu E^\mu. \quad (86)$$

Considering that S and E are given respectively by (17) and (81), Eq. (86) implies

$$S(t) + E(t) = S(0) + E(0). \quad (87)$$

Now we impose that the theory is stable and that all the perturbations (which conserve the original value of the

constants of motion of the fluid) decay to zero as $t \rightarrow +\infty$, implying

$$S(+\infty) = S_{\text{eq}} \quad E(+\infty) = 0. \quad (88)$$

The second condition results from the fact that E^μ is quadratic in the perturbation and therefore is exactly zero in equilibrium. The quantity S_{eq} is simply the value of the entropy in the unperturbed equilibrium state. If we plug these two conditions in (87), we obtain

$$S_{\text{eq}} = S(0) + E(0), \quad (89)$$

which plugged again into (87) implies

$$E = S_{\text{eq}} - S. \quad (90)$$

Since this relation holds for any initial (dynamically accessible) small perturbation, we have proved that the functional E is nothing but the second-order correction to the entropy in the deviations from equilibrium. The stability condition (83), then, implies

$$S_{\text{eq}} \geq S, \quad (91)$$

which is the requirement that the entropy is maximum in the equilibrium state and is, therefore, a Lyapunov function.

D. Example: Stable heat conduction

It is interesting to verify how the second-order terms in the entropy current counterbalance the explosion of the first-order theories with a concrete example. We consider again the Eckart fluid introduced in subsection III A, but we replace the entropy current (25) with the Israel-Stewart prescription

$$s^\mu = s u^\mu + \frac{1}{\Theta} q^\mu - \frac{\beta_1 q^\nu q_\nu}{2\Theta} u^\mu. \quad (92)$$

The stability condition (i.e., the condition to have $E \geq 0$) obtained by [18] for this model reads

$$\beta_1 > \frac{1}{\rho + P}. \quad (93)$$

Let us verify that this makes the state $v = 0$ (i.e., the homogeneous perfect fluid configuration) a local maximum of the entropy, as discussed in Sec. III A.

Since the stress-energy tensor is the same, the steps which lead us to Eq. (42) are unchanged, apart from the fact that we need to add to s^0 the second order contribution, obtaining

$$s^0 = \tilde{s} \left(1 + \frac{v^2}{2} \right) - \frac{\beta_1 q^1 q^1}{2\Theta}. \quad (94)$$

We have made the replacement $q^\nu q_\nu = q^1 q^1$ because $-q^0 q^0$ is a fourth order term, see Eq. (31). The condition that the entropy is maximum in equilibrium reads

$$\tilde{s} \geq s^0, \quad (95)$$

which, with the aid of (41), implies

$$\beta_1 > \tilde{s} \Theta \left(\frac{4}{3} \mathcal{E} \right)^{-2}. \quad (96)$$

All the quantities appearing in the inequality above are background terms [as can be seen from (94)], thus they can be evaluated at $v = 0$, which, from (40), we know to be characterized by the conditions

$$\rho = \mathcal{E} \quad s = \tilde{s}. \quad (97)$$

However, from (11) and (13), we also have the chain of identities

$$s\Theta = \frac{4}{3}\rho = \rho + P. \quad (98)$$

With these conditions it is possible to prove the equivalence of (96) with (93). We have thus verified that the condition of dynamic stability of the Israel-Stewart theory coincides with the condition of maximum entropy in equilibrium.

E. The problem of the instability for large deviations

Equation (93) is the condition for \tilde{s} to be the maximum value of s^0 for small deviations from equilibrium. However, it is not guaranteed that this maximum is absolute. In fact, Hiscock and Lindblom [42] have shown that, for sufficiently large deviations from equilibrium, the system can still admit runaway solutions. Let us analyze the behavior of s^0 , given in (92), for an arbitrarily large v .

We can split the entropy density as

$$s^0 = s_I^0 + s_{II}^0, \quad (99)$$

where s_I^0 is the zeroth + first order contribution, given by Eq. (37), and

$$s_{II}^0 = -\frac{\beta_1 \gamma}{2\Theta} (q^1 q^1 - q^0 q^0) \quad (100)$$

is the second-order contribution. To make a parametric study, we take as a prescription for β_1 the generic expression

$$\beta_1 = \frac{b}{\rho + P}, \quad (101)$$

where $b \geq 0$ is a free constant coefficient. With calculations which are analogous to those made in Sec. III one can obtain the expression

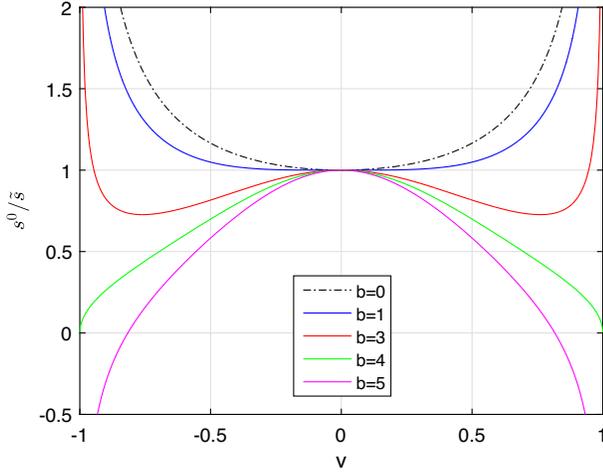


FIG. 3. The normalized Israel-Stewart entropy per unit volume s^0/\bar{s} as a function of the velocity v , as given by Eq. (103), for $b = 0, 1, 3, 4, 5$. We see that the local stability condition $b > 1$ obtained by Hiscock and Lindblom [18] corresponds to the requirement that the equilibrium state $v = 0$ is a local maximum of the entropy. However, if we want to ensure the global stability, this maximum must be a global maximum and we need to impose the stronger requirement $b \geq 4$.

$$\frac{s_{II}^0}{\bar{s}} = -\frac{b}{2}v^2(1-v^2)^{-1/2}(1+v^2)^{-5/4}\left(1-\frac{v^2}{3}\right)^{-3/4}. \quad (102)$$

Plugging this formula into (99) and recalling (37), we obtain

$$\frac{s^0}{\bar{s}} = \left(1 + \frac{2-b}{2}v^2\right)(1-v^2)^{-1/2}(1+v^2)^{-5/4}\left(1-\frac{v^2}{3}\right)^{-3/4}. \quad (103)$$

In Fig. 3 we show how the profile of s^0/\bar{s} varies for different values of b . For $b < 1$ the stability condition (93) for small deviations from equilibrium is not fulfilled and the state $v = 0$ is a minimum. The situation $b = 1$ is the case in which the second order expansion of s^0 around $v = 0$ is zero. We see that the next nonvanishing order in v , the fourth order, is still positive: the perfect fluid state is still a minimum, and therefore is unstable. For $1 < b < 4$ the second order term of the expansion is negative, thus $v = 0$ is a maximum. However, it is a local maximum, because for large v^2 the positive divergence of s_I^0 still dominates over the negative contribution of s_{II}^0 . For the critical value $b = 4$ the two divergences compensate each other exactly and we have $s^0 \rightarrow 0$ as $v \rightarrow \pm 1$. For $b \geq 4$ the theory is stable for any homogeneous deviation from equilibrium and $v = 0$ is likely to be the absolute maximum of the entropy (to be certain one should also make a study involving non-homogeneous configurations).

Therefore, we have shown that the condition (93) is not enough to guarantee that the entropy is a Lyapunov

function over the whole configuration space, but there is an interval of values of the parameter b ($1 < b < 4$) in which the homogeneous perfect fluid is only a point of local maximum.

Again, we remark that to study the stability of the theory there is no need to know the details of the equations of motion of the fluid. In fact, we have not invoked their hyperbolic nature or their telegraph-type form. The only property needed is to know that they are constructed in a way to ensure that the entropy current (92) has a non-negative divergence.

F. The effect of the particle conservation

In their analysis, Hiscock and Lindblom [42] consider the case $b = 5$ and, even if they are working with an ultrarelativistic gas and they impose $b > 4$, they still find that the model is unstable for large deviations from equilibrium. The key difference with our case is that they have a conserved particle number, while we are working at zero chemical potential. The conservation of the particle number has the effect of increasing the instability for large v .

To show this, we first note that (73) implies in the homogeneous case

$$n^0 = \gamma n = \text{const.} \quad (104)$$

Now, since equation (34) is a consequence only of the structure of the energy-momentum tensor and not of the equation of state, it holds also when the particle number is conserved. Therefore, assuming the ideal gas law (valid in the nondegenerate limit)

$$P = n\Theta, \quad (105)$$

we obtain

$$\Theta = \gamma \frac{\mathcal{E}}{n^0} \frac{1+v^2}{3-v^2}, \quad (106)$$

which is Eq. (15) of [42]. This relation implies that the temperature diverges when $v^2 \rightarrow 1$. This is the key difference with respect to our case at zero chemical potential, because in that case all the rest-frame thermodynamic quantities remain finite also when the fluid approaches the speed of light.

To see the implications on the stability, we insert (106) into (100). As a result, the Lorentz factors cancel out and s_{II}^0 is now finite even for $v^2 = 1^-$. The second-order contribution to the entropy is, therefore, no longer able to compensate the divergence of the first-order term s_I^0 and the fluid is unstable for any b when sufficiently large deviations from equilibrium are considered, in agreement with [42].

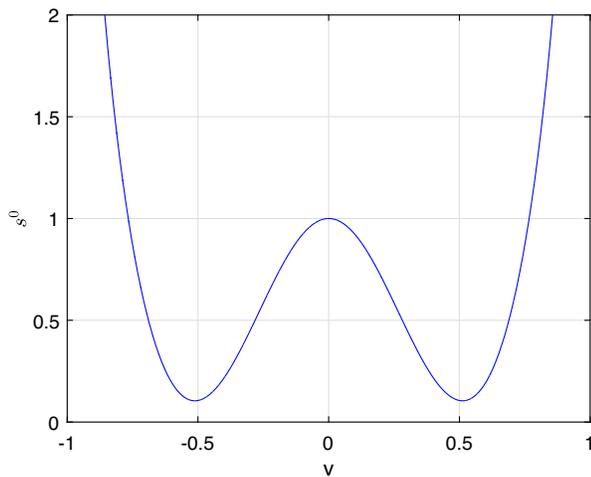


FIG. 4. Israel-Stewart entropy density s^0 , given in Eq. (92), for a nondegenerate ultrarelativistic homogeneous heat-conducting gas with conserved particle number, zero total momentum and $\beta_1 = 5/(\rho + P)$. This is the case considered by Hiscock and Lindblom [42]. We have imposed $\mathcal{E} = n^0 = 1$ and we have set the units in a way that $s^0(0) = 1$. As can be seen, $v = 0$ is only a local maximum of the entropy density and this is the origin of the instability for large deviations from equilibrium.

To complete the comparison, in Fig. 4 we show the profile of $s^0(v)$ for the fluid considered by [42]. The analytical formula is not reported here, but it can be easily obtained by following analogous steps to those which lead to (103), knowing that the equilibrium entropy per particle associated with the equation of state (105) is

$$\frac{s}{n} = -\ln\left(\frac{n}{\Theta^3}\right) + \text{const.} \quad (107)$$

We see from figure 4 that $s^0(v)$ has a minimum in $|v| = v_c := 0.51188$, which is the critical velocity at which the fluid becomes unstable. Now the thermodynamic origin of the instability of Israel-Stewart is clear: for $|v| < v_c$ the direction of positive entropy growth points toward $v = 0$, thus $|v|$ decreases. On the other hand, for $v > v_c$, the speed has to increase to enforce the validity of the second law. Thus, the bifurcation found by [42] at $|v| = v_c$ is a direct result of the entropy profile given in Fig. 4 and of the strict obedience of the system to the second law of thermodynamics.

VII. FIRST-ORDER THEORIES IN MORE GENERAL FRAMES

There is increasing attention on new first-order theories which have been proved to be causal and stable, if the transport coefficients are appropriately tuned [9,20]. The idea is to use alternative hydrodynamic frames to both Eckart's and Landau-Lifschitz [19,21]. In this final section we show that the stability of these theories is enforced by

allowing for small violations of the second law of thermodynamics.

A. A model for bulk viscosity

To simplify our analysis we restrict our attention to a purely bulk-viscous fluid. The stress-energy tensor we consider is

$$T^{\mu\nu} = (\rho + P + \mathcal{A})u^\mu u^\nu + P g^{\mu\nu}. \quad (108)$$

The equilibrium perfect-fluid energy-momentum tensor is still the one we presented in Sec. II B. The deviations from equilibrium, instead of being located in a viscous stress contribution, are given by a correction \mathcal{A} to the internal energy of the fluid,

$$\mathfrak{T}^{\mu\nu} = \mathcal{A}u^\mu u^\nu. \quad (109)$$

Instead of the standard dependence of the viscous stresses on $\nabla_\mu u^\mu$ only, the nonequilibrium correction to the energy density is assumed to be

$$\mathcal{A} = \chi_1 \frac{u^\mu \nabla_\mu \rho}{\rho + P} + \chi_2 \nabla_\mu u^\mu. \quad (110)$$

The transport coefficients χ_1 and χ_2 are functions of ρ . The entropy current is assumed to have the form

$$s^\mu = \left(s + \frac{\mathcal{A}}{\Theta}\right)u^\mu, \quad (111)$$

which, then, implies that the nonequilibrium correction is

$$\sigma^\mu = \frac{\mathcal{A}}{\Theta}u^\mu. \quad (112)$$

By using the conservation of the energy-momentum ($\nabla_\mu T^{\mu\nu} = 0$), it can be shown that the equation for the entropy production reads

$$\Theta \nabla_\mu s^\mu = -\mathcal{A} \frac{u^\mu \nabla_\mu \Theta}{\Theta}. \quad (113)$$

It is now immediate to see that to ensure the validity of the second law (16) for any fluid configuration we need to impose

$$\chi_1 \leq 0 \quad \chi_2 = 0. \quad (114)$$

However, following the argument of Kovtun [9], one may consider that, since (111) is just an approximate formula for the entropy current, its four-divergence needs to be only approximately non-negative. In particular, if one considers that the deviations from equilibrium are small, then the effect of the dissipative terms on the hydrodynamic

evolution are small and we can impose the perfect-fluid relation

$$u^\mu \nabla_\mu \rho \approx -(\rho + P) \nabla_\mu u^\mu \quad (115)$$

as approximately satisfied. Therefore, using the definition (110), at the lowest order we can replace \mathcal{A} in (113) with the approximate expression

$$\mathcal{A} \approx (\chi_1 - \chi_2) \frac{u^\mu \nabla_\mu \rho}{\rho + P}. \quad (116)$$

The constraint of non-negative entropy production, then, requires only

$$\chi_1 - \chi_2 \leq 0. \quad (117)$$

As can be seen, this is a much weaker constraint than (114). Indeed, the only way to make the theory stable is to impose that the two separate conditions (114) are both violated, while keeping (117) fulfilled. In particular, Bemfica *et al.* [20] have derived the stability conditions

$$\chi_2 > \chi_1 > 0. \quad (118)$$

Thus we see that a necessary condition of stability is that the second law is not exactly respected, but it holds only as an approximation (see also [22]). Our goal in the next subsection is to explain why this is the case.

B. Thermodynamic analysis

Similarly to what we did in the previous sections, we consider a homogeneous portion of the fluid and we impose for simplicity $\mathcal{P}^j = 0$. Then it is immediate to see that there is no way for the fluid to change its velocity. This implies that the relevant degrees of freedom of the fluid element are only two (ρ and \mathcal{A}) and we have only one relevant constraint (\mathcal{E}). Again, we are dealing with a 1D manifold of dynamically allowed states. We parametrize it with \mathcal{A} and we use the constraint equation (28) to write ρ as a function of \mathcal{A} :

$$\rho = \mathcal{E} - \mathcal{A}. \quad (119)$$

From (111), we immediately obtain

$$s^0 = s + \frac{\mathcal{A}}{\Theta}. \quad (120)$$

With calculations analogous to those made in the previous sections we can obtain the formula for the entropy density as a function of the parameter of the curve:

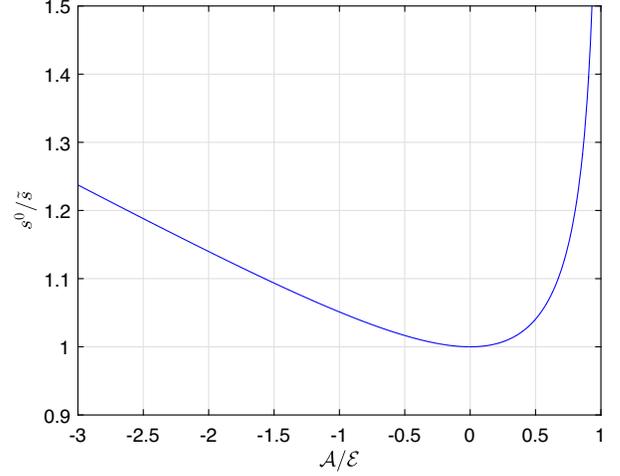


FIG. 5. The normalized entropy per unit volume s^0/\tilde{s} as a function of the dimensionless parameter \mathcal{A}/\mathcal{E} as given by Eq. (121). For $\mathcal{A} = 0$ the fluid is in local thermodynamic equilibrium. We see that, also in this case, the entropy, restricted to the manifold of the dynamically allowed states, is not maximal in equilibrium. Thus if the second law was exactly respected, the theory of Bemfica *et al.* [20] would necessarily be unstable.

$$s^0 = \tilde{s} \left(1 - \frac{\mathcal{A}}{4\mathcal{E}}\right) \left(1 - \frac{\mathcal{A}}{\mathcal{E}}\right)^{-1/4}. \quad (121)$$

The graph of this function is shown in Fig. 5.

Also in this case the perfect fluid state (given by $\mathcal{A} = 0$) is a saddle point of the entropy and not its maximum. Therefore, if we enforce the entropy production to be strictly non-negative, a perturbation which makes the system start with $\mathcal{A} \neq 0$ is not allowed to decay but can only grow. This tells us that it is necessary to violate the second law of thermodynamics to ensure that the perfect fluid state is a stable equilibrium configuration.

Let us study more in detail the evolution of small homogeneous perturbations around $\mathcal{A} = 0$. We consider perturbations of the generic hydrodynamic function f to have the form

$$\delta f e^{\Gamma t}, \quad (122)$$

where δf is an infinitesimal constant, and we impose that these perturbations preserve the constraints (thus they are dynamically allowed). From (110) we can write the perturbation \mathcal{A} at the first order as

$$\delta \mathcal{A} = \frac{3\chi_1}{4\mathcal{E}} \Gamma \delta \rho. \quad (123)$$

Note that the perturbation of $\nabla_\mu u^\mu$ is exactly zero because the fluid remains at rest. If we vary (119) we obtain $\delta \rho = -\delta \mathcal{A}$, which, plugged in the equation above, gives

$$\Gamma = -\frac{4\mathcal{E}}{3\chi_1}. \quad (124)$$

The condition for the perturbation to decay is $\Gamma < 0$, which implies $\chi_1 > 0$. Thus we recover one of the two stability conditions (118) found by Bemfica *et al.* [20]. On the other hand, if we want the second law to be strictly satisfied, then we need impose $\Gamma \geq 0$, which in turn implies $\chi_1 < 0$. In this case we recover the first condition of (114).

The reason why in this configuration there is such a strong contradiction between the stability criteria and the second law is that we have constructed the system in a way to enforce $\nabla_\mu u^\mu = 0$, while imposing $u^\mu \nabla_\mu \rho \neq 0$. Thus, the approximate relation (115) does not hold for this initial condition and (116) is no longer valid.

We remark that the amount of entropy which is annihilated is a second order in \mathcal{A} . However, the entropy current (111) is only an approximation to the first order in \mathcal{A} of the physical entropy current. Then, as we explained in Sec. V, this violation should not be considered a reason of concern, but, instead, represents the fundamental origin of the success of the theory.

VIII. CONCLUSIONS

In relativity, spatial gradients become linear combinations of derivatives in both space and time when one moves from one reference frame to the other. The number of degrees of freedom of the relativistic models of dissipation is, then, larger with respect to the Newtonian counterpart. This implies that the state-space has a bigger dimension and hosts a wide set of configurations which do not have a Newtonian analogue. We have shown that the instability of the Eckart [11] and Landau and Lifshitz [12] theories arises from the fact that it is always possible to find a dynamically allowed path in this extended state space along which the total entropy grows with no bound.

This fact tells us that the instability of both Eckart and Landau-Lifshitz theories has a purely thermodynamic nature: the assumed equilibrium state of the fluid (such as, for example, an homogeneous perfect fluid configuration) is not the maximum of the total entropy, but only a saddle point. The unstable modes evolve along the directions in the state space in which the entropy takes larger values with respect to the value assumed in the (supposed) equilibrium state. This also shows that the reason why these fluids do not obey the Onsager regression hypothesis (as has been pointed out by Garcia-Perciante *et al.* [36]) is that an equilibrium state does not exist and the entropy is not a Lyapunov function of the system.

We have then proved that the conditions for stability obtained by Hiscock and Lindblom [18] are the requirements for the second-order theory to produce a system with a maximum entropy state (for small deviation from equilibrium). The second-order contributions act in a way to compensate the explosion of the first-order theory. In addition, we have seen that the same study can also be used to prove the possible instability of the Israel and Stewart [6] theory for large deviations from equilibrium. In

fact, we have verified that the critical speed $v_c = 0.51188$ found by Hiscock and Lindblom [42] at which a homogeneous heat-conducting ultrarelativistic fluid becomes unstable marks a saddle point of the entropy above which the latter starts growing with no bound, producing the instability. Finally, we have shown that the first-order theory of Bemfica *et al.* [20] does not restore the concavity of the entropy, but deals with the stability problem by allowing for small violations of the second law. In this kind of first-order theories the strict obedience to the second-principle would lead again to the (nonphysical) explosion of small perturbations.

The main message of the present analysis is that, assuming a first-order expansion of the entropy current, one is selectively removing second-order contributions to the total entropy of the fluid. In this way, the concavity properties of the entropy are altered and, as a consequence, the absolute maximum is typically converted into a (nonphysical) saddle point. The instability, then, arises when the second law is imposed, enforcing the growth of the approximated entropy at all the orders. In this way, the thermodynamic principle which originally was ensuring the Lyapunov stability of the system is converted into the main source of instability, pushing the system along the non-physical growing branches which depart from the equilibrium state.

There are only two possible solutions to this problem. The first is to retain all the second-order contributions (which leads to the higher order formulations like Israel and Stewart [6] and Carter [8]) the second is to break the second law of thermodynamics at the second order (which leads to the first-order theories in more general frames proposed by Bemfica *et al.* [20]).

Finally, our discussion also clarifies that in relativity the approach of extended irreversible thermodynamics [34,35], which promotes the dissipative terms to degrees of freedom, is a mathematical necessity. Even the first-order theories, which are not explicitly designed according to this philosophy, implicitly contain this assumption, which manifests itself as soon as the fluid is set into motion. Once this general fact is accepted, all the interpretative difficulties disappear and thermodynamics rules once again.

ACKNOWLEDGMENTS

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APPENDIX: INSTABILITY OF THE DIFFUSION EQUATION IN RELATIVITY

In this appendix we study the instability of the diffusion equation in special relativity. Despite its simplicity, this example provides physical intuition of how, changing reference frame, one might produce unexpected

instabilities. A detailed analysis of the mathematical aspects of the problem can be found in Kostädt and Liu [33], who proved that all the problems concerning the diffusion equation in relativity arise from its ill-posedness in the boosted frame. They also showed that the instability mechanisms of Landau and Lifshitz [12] are formally identical to those of the diffusion equation, ensuring the generality of the results of the present appendix.

1. The role of the relativity of simultaneity

Consider a one-dimensional medium whose temperature field Θ obeys the diffusion equation (in rest frame of the medium)

$$\frac{\partial \Theta}{\partial t} = \mathcal{D} \frac{\partial^2 \Theta}{\partial x^2}, \quad (\text{A1})$$

where \mathcal{D} is, for simplicity, a constant. We ignore for the moment the issues related with causality, accepting the idea that signal propagation in this medium can be superluminal. We are interested in the evolution of Θ , as seen by an observer who is moving with velocity $v \neq 0$ with respect to the medium. The associated Lorentz transformation is

$$t' = \gamma(t - vx) \quad x' = \gamma(x - vt), \quad (\text{A2})$$

with

$$\gamma = \frac{1}{\sqrt{1 - v^2}}, \quad (\text{A3})$$

implying

$$\frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right) \quad \frac{\partial}{\partial x} = \gamma \left(\frac{\partial}{\partial x'} - v \frac{\partial}{\partial t'} \right). \quad (\text{A4})$$

In the boosted frame, Eq. (A1) now reads

$$\frac{\partial \Theta}{\partial t'} - v \frac{\partial \Theta}{\partial x'} = \mathcal{D} \gamma \left(\frac{\partial^2 \Theta}{\partial x'^2} - 2v \frac{\partial^2 \Theta}{\partial t' \partial x'} + v^2 \frac{\partial^2 \Theta}{\partial t'^2} \right). \quad (\text{A5})$$

Note that Eq. (A1) was of the first order in the reference frame of the medium, but it becomes of the second order in any other reference frame, due to the presence of the third term in the right-hand side of (A5). Therefore, if in the rest-frame of the medium the state is entirely specified once we know the value of Θ everywhere at the initial time, in any other frame we need to know both Θ and $\partial \Theta / \partial t'$. The principle of relativity has enlarged the state-space, increasing the number of degrees of freedom of the system, as we discussed in Sec. II B.

The origin of the problem is the relativity of simultaneity [43], namely the fact that events which are simultaneous in a given reference frame may not be simultaneous in another one. In fact, if there was an absolute notion of

simultaneity, which would imply that $t' = t'(t)$, we would have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \Big|_t \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x} \Big|_t \frac{\partial}{\partial t'} = \frac{\partial x'}{\partial x} \Big|_t \frac{\partial}{\partial x'} \quad (\text{A6})$$

and the second derivative in time in equation (A5) would not appear.

In some astrophysical contexts [44], diffusion-type models for relativistic dissipation are included in numerical simulations assuming that in the reference frame considered in the simulation (which does not necessarily coincide with the rest-frame of the medium) the evolution can be approximated as quasistatic. This assumption is then used to neglect the term $\partial_t^2 \Theta$ in (A5). The resulting system is, then, not structurally different from a Newtonian model (it has the same number of degrees of freedom) and stability can be restored.

2. Stability analysis

Let us study the evolution of homogeneous configurations in the frame which is moving with respect to the medium. Equation (A5) reduces to

$$\frac{\partial \Theta}{\partial t'} = \mathcal{D} \gamma v^2 \frac{\partial^2 \Theta}{\partial t'^2}, \quad (\text{A7})$$

whose general solution is

$$\Theta(t') = \Theta_0 + \frac{\dot{\Theta}_0}{\Gamma_+} (e^{\Gamma_+ t'} - 1), \quad (\text{A8})$$

where we have defined

$$\Gamma_+ = \frac{1}{\mathcal{D} \gamma v^2} > 0. \quad (\text{A9})$$

The space of the initial conditions is determined by two parameters (Θ_0 and $\dot{\Theta}_0$), instead of only one. Furthermore, we see that, whenever $\dot{\Theta}_0 \neq 0$, the solution diverges for large times. Hence, we have verified that the existence of unstable solutions arises directly from the possibility of setting the time-derivative of Θ freely and is, therefore, a pure consequence of the relativistic extension of the state-space.

To understand how the instability can develop in a boosted reference frame, while it does not appear in the frame of the medium, let us examine solutions given by initial conditions of the type

$$\dot{\Theta}_0 = \Gamma_+ \Theta_0. \quad (\text{A10})$$

Going to the medium rest-frame, using the transformation rule (A2), we immediately see that these solutions have the form

$$\Theta(t, x) = \Theta_0 e^{\Gamma_+ \gamma(t-vx)}, \quad (\text{A11})$$

which is obviously a solution of (A1). We see that along surfaces at constant time the space dependence of Θ is

$$\Theta(t, x) \propto e^{-\Gamma_+ \gamma vx}. \quad (\text{A12})$$

This means that (assuming $v > 0$ for definiteness) the unstable solutions in the boosted frame correspond to configurations in the frame of the medium in which an infinite amount of energy is shifting uniformly to the right in the spacetime diagram, coming from $x = -\infty$. This again shows how the relativity of simultaneity, making even the notion of homogeneity frame-dependent, is a key ingredient to make the instability possible.

It is interesting to note that in the Newtonian limit ($v \rightarrow 0$) the growth rate diverges (and does not go to zero as one might intuitively think), $\Gamma_+ \rightarrow +\infty$. This happens also in the theories of [11,12]. The reason is that, since the Newtonian theory has less degrees of freedom than the relativistic one, we cannot obtain it just by taking the limit of the equations of motion, but we also need to make a particular choice for the initial conditions.

This can be understood by considering again Eq. (A7), evaluated at $t' = 0$:

$$\frac{\partial^2 \Theta}{\partial t'^2}(0) = \frac{\dot{\Theta}_0}{\mathcal{D}\gamma v^2}. \quad (\text{A13})$$

If we take the limit $v \rightarrow 0$, while keeping $\dot{\Theta}_0$ fixed and finite, we find

$$\frac{\partial^2 \Theta}{\partial t'^2}(0) \rightarrow \infty. \quad (\text{A14})$$

We, thus, obtain a fast diverging solution. However, we know that in the Newtonian limit $\dot{\Theta}_0$ cannot be set arbitrarily, but must be zero, as predicted by (A1). Therefore, to obtain the Newtonian theory, we need to send $v \rightarrow 0$, while selecting the initial condition for $\dot{\Theta}_0$ directly from the Newtonian equation (A1). This gives the expected result:

$$\frac{\partial^2 \Theta}{\partial t'^2}(0) = 0. \quad (\text{A15})$$

3. The Cattaneo hyperbolic model

The Cattaneo equation [45] is a modified diffusion equation that ensures finite signal propagation speed. This equation, as shown by Israel and Stewart [6], arises naturally from a relativistic thermodynamic approach, and includes a relaxation term with a timescale $\tau > 0$,

$$\tau \frac{\partial^2 \Theta}{\partial t^2} + \frac{\partial \Theta}{\partial t} = \mathcal{D} \frac{\partial^2 \Theta}{\partial x^2}. \quad (\text{A16})$$

It is known (see e.g., [33]) that the foregoing equation (which now is of the second order in the time-derivative even in the rest-frame) admits a signal propagation which cannot exceed the speed

$$c_{II} = \sqrt{\frac{\mathcal{D}}{\tau}}, \quad (\text{A17})$$

called second-sound speed. It is, then, clear that the theory is compatible with causality requirements if and only if

$$c_{II} \leq 1. \quad (\text{A18})$$

We can easily verify that this modification solves also the stability problems. In fact, from (A4), we find

$$\tau \frac{\partial^2 \Theta}{\partial t'^2} = \tau \gamma^2 \left(\frac{\partial^2 \Theta}{\partial t'^2} - 2v \frac{\partial^2 \Theta}{\partial t' \partial x'} + v^2 \frac{\partial^2 \Theta}{\partial x'^2} \right). \quad (\text{A19})$$

The equation for the homogeneous solutions (A7), then, becomes

$$\tau \gamma \frac{\partial^2 \Theta}{\partial t'^2} + \frac{\partial \Theta}{\partial t'} = \mathcal{D} \gamma v^2 \frac{\partial^2 \Theta}{\partial t'^2}, \quad (\text{A20})$$

whose general solution is

$$\Theta(t') = \Theta_0 + \frac{\dot{\Theta}_0}{\Gamma_-} (e^{\Gamma_- t'} - 1), \quad (\text{A21})$$

with

$$\Gamma_- = \frac{1}{\gamma(\mathcal{D}v^2 - \tau)}. \quad (\text{A22})$$

The stability requirement is $\Gamma_- < 0$, which (imposed for every $v^2 < 1$) implies (A18).

4. Connection between causality and stability

Olson and Hiscock [17], Hiscock and Lindblom [18] have shown that, in Israel-Stewart-type theories, stability and causality are essentially equivalent. As we saw in the foregoing subsection, the Cattaneo equation is no exception and we can use it to give a simple physical intuition of this connection.

Let us consider a temperature profile of the form

$$\Theta = \Theta_0 e^{\Gamma \gamma(t-vx)} \quad \Theta_0 > 0. \quad (\text{A23})$$

Clearly, if $\Gamma > 0$ the underlying theory of which this profile is solution is unstable in the boosted frame.

Neglecting overall additive constants, the energy per unit length (measured in the rest-frame of the medium) is

$$\rho = c_v \Theta, \quad (\text{A24})$$

where c_v is the specific heat (note that this is one of the approximations one needs to invoke in order to derive (A1)). Now, let us define the function

$$\mathcal{E}(t) := \int_t^{+\infty} \rho(x, t) dx, \quad (\text{A25})$$

which measures the amount of energy contained in the half-line $x > t$ at the time t . Since the half-line $x > t_1$ is the causal past of $x > t_2$ (for $t_1 < t_2$), then, if the theory is causal, we need to have $\mathcal{E}(t_1) \geq \mathcal{E}(t_2)$, because no energy can be transferred to a region from outside its past

light-cone. Therefore, if the theory is causal, it must be true that

$$\frac{d\mathcal{E}}{dt} \leq 0. \quad (\text{A26})$$

On the other hand, it is immediate to verify that

$$\mathcal{E}(t) = \mathcal{E}(0)e^{\Gamma(1-v)t}, \quad (\text{A27})$$

which proves that if the theory is unstable ($\Gamma > 0$), then it is not causal ($d\mathcal{E}/dt > 0$).

This argument shows that in a causal theory there is not enough energy to develop instabilities of the form (A23). In fact, it is necessary to allow for superluminal transport of energy to transfer the energy from $x = -\infty$ to the bulk of the system sufficiently fast (in the rest-frame of the medium) to produce the instability.

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Coauthor Statement

Article under discussion:

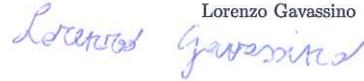
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Declaration of Lorenzo Gavassino (first author):

I, Lorenzo Gavassino, hereby declare that my contribution to the aforementioned research article amounts to 90%. I had the original idea, decided how to address the problem, and performed the mathematical analysis entirely. I also created the plots and wrote the draft.

Lorenzo Gavassino



Declaration of Marco Antonelli (second author):

I, Marco Antonelli, hereby declare that my contribution to the aforementioned research article amounts to 5%. I contributed to the preparation of the draft.

Marco Antonelli



Declaration of Brynmor Haskell (third author):

I, Brynmor Haskell, hereby declare that my contribution to the aforementioned research article amounts to 5%. I contributed to the preparation of the draft.

Brynmor Haskell



Chapter 7

Applying the Gibbs stability criterion to relativistic hydrodynamics

Letter

Applying the Gibbs stability criterion to relativistic hydrodynamics

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Abstract

The stability of the equilibrium state is one of the crucial tests a hydrodynamic theory needs to pass. A widespread technique to study this property consists of searching for a Lyapunov function of the linearised theory, in the form of a quadratic energy-like functional. For relativistic fluids, the explicit expression of such a functional is often found by guessing and lacks a clear physical interpretation. We present a quick, rigorous and systematic technique for constructing the functional of a generic relativistic fluid theory, based on the maximum entropy principle. The method gives the expected result in those cases in which the functional was already known. For the method to be applicable, there must be an entropy current with non-negative four-divergence. This result is an important step towards a definitive resolution of the major open problems connected with relativistic dissipation.

Keywords: relativistic hydrodynamics, curved space–time, stability, thermodynamics, entropy, viscosity, chemistry

 Supplementary material for this article is available [online](#)

(Some figures may appear in colour only in the online journal)

1. Introduction

Recent years have seen an explosion of new dissipative hydrodynamic theories, as fluid descriptions are applied to different fields, ranging from heavy ion collisions [1], to neutron star physics [2] and cosmology [3]. The demand for new theories comes from the inadequacy of simple fluids to account for the complexity of real systems. For example, current theories for viscosity [4, 5] fail to describe the initial transient of strongly interacting quantum field theories [6, 7]. Furthermore, cold neutron-star matter is a superfluid-normal mixture, which

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requires multi-fluid modelling [8–12]. Even less exotic astrophysical systems (such as accretion disks and jets) cannot accurately be described using simple fluids, due to the presence of a magnetic field, a radiation field and two-temperature effects [13–15]. Combining these features with causal dissipation leads to completely new theories, e.g. [16]. Finally, hot dense matter in supernovae and neutron-star mergers is a reacting mixture, with reaction time-scales comparable with the hydrodynamic time-scale [17–19]. This requires us to revise our understanding of causal bulk viscosity [20].

As more and more complex theories are proposed, it is of central importance to be able to predict if the theory that one is building is truly dissipative (i.e. if the fluid exhibits a tendency to evolve towards thermodynamic equilibrium) or if the non-equilibrium degrees of freedom undergo a non-physical spontaneous explosion, as in the case of the theories of Eckart and Landau–Lifshitz [21, 22]. This criterion of *stability of the equilibrium*, according to which states that are initially close to global *thermodynamic* equilibrium [23–27] remain close to it, constitutes the most fundamental reliability test of a dissipative theory [28]. Unfortunately, verifying this property with the current techniques is usually complicated and the physical interpretation of the stability conditions is often not transparent [29–32]. In fact, the calculation strongly depends on the details of the hydrodynamic equations: adding a new coupling or slightly modifying the physical setting might force one to start over the whole stability analysis [32–34]. More importantly, one would like to be able to test the stability of any possible thermodynamic equilibrium state (at rest or in motion, rotating or non-rotating, with or without a strong gravitational field) at once, while often (when the theory becomes too complicated) the calculation is specialised to homogeneous equilibria in a Minkowski background [34–39].

On the other hand, the theory of *thermodynamic* stability has a long history, which goes back to Gibbs [40]. The idea of Gibbs was simple: since the entropy cannot decrease, the equilibrium state of an isolated system is stable if any (physically allowed) perturbation results in a decrease in entropy. In other words, the entropy should be maximum in equilibrium, to ensure Lyapunov stability [41]. Here, we apply this principle to relativistic hydrodynamics, presenting a technique to build, directly from the constitutive relations of a generic fluid, a quadratic Lyapunov functional, whose positive-definiteness implies stability. Below we outline the methodology and we give a couple of examples and applications. We adopt the signature $(-, +, +, +)$ and work in units $c = k_B = 1$.

2. The stability criterion

It is crucial for our method that we can associate to the fluid a symmetric stress–energy tensor T^{ab} and an entropy current s^a which obey the conditions

$$\nabla_a T^{ab} = 0 \quad \nabla_a s^a \geq 0 \quad (1)$$

as *exact* mathematical constraints. The remaining details of the field equations (such as the exact value of the entropy production rate) are irrelevant and do not play any role in the method, provided that (1) are respected. Here we assume, for illustrative purposes, that there is a single conserved current N^a , such that $\nabla_a N^a = 0$, but the method can be straightforwardly generalised to fluids with an arbitrary number of conserved currents. We assume that the fluid is immersed in a test spacetime (which plays the role of a fixed background), having one and only one Killing vector field K^a , which is everywhere time-like future-directed. If the fluid has a finite spatial extension, then, assigned a space-like Cauchy 3D-surface Σ , the three integrals

$$\{N, U, S\} = \int_{\Sigma} \{-N^a, T^{ab} K_b, -s^a\} d\Sigma_a \quad (2)$$

are finite and represent the total particle number, energy and entropy of the fluid. Given the aforementioned assumptions, N and U are conserved (i.e. they do not depend on Σ), while

$$S[\Sigma'] \geq S[\Sigma] \quad (3)$$

whenever Σ' is in the future of Σ . We also need to have a selection of the macroscopic fields φ_i which carry information about the local state of the fluid (e.g. for the perfect fluid one may take the fluid velocity, the temperature and the chemical potential) and the constitutive relations:

$$T^{ab} = T^{ab}[\varphi_i] \quad s^a = s^a[\varphi_i] \quad N^a = N^a[\varphi_i]. \quad (4)$$

The method works as follows: we consider two solutions of the (in principle unknown) hydrodynamic equations, which are close to each other,

$$\varphi_i \quad \text{and} \quad \varphi_i + \delta\varphi_i, \quad (5)$$

and we define the variation of any observable \mathcal{A} as the *exact* difference¹

$$\delta\mathcal{A} := \mathcal{A}[\varphi_i + \delta\varphi_i] - \mathcal{A}[\varphi_i]. \quad (6)$$

The configuration φ_i is our candidate global thermodynamic equilibrium state, while $\delta\varphi_i$ is a deviation from equilibrium which should decay to zero for large times (if the theory is dissipative and stable). For this to be possible, we must impose that the integrals of motion U and N have exactly the same value in the two states, namely

$$\delta N = \delta U = 0, \quad (7)$$

otherwise $\varphi_i + \delta\varphi_i$ would asymptotically relax to an equilibrium state which is different from φ_i .² If there are additional constants of motion, like e.g. a superfluid winding number [12], these need to be treated on the same footing as N and U .

Now we only need to take two steps:

- (a) We truncate all the differences $\delta\mathcal{A}$ to the first order in $\delta\varphi_i$ and we impose the stationarity condition $\delta S = 0$ for any possible choice of $\delta\varphi_i$ compatible with (7). This procedure *defines* the thermodynamic equilibrium state φ_i and identifies it completely. At this stage (and only for step (a)), one may deal with the constraint (7) using some Lagrange multipliers α and β , rediscovering the covariant Gibbs relation [4, 23, 24]

$$\int_{\Sigma} (\delta s^a + \alpha \delta N^a + \beta K_b \delta T^{ab}) d\Sigma_a = 0 \quad (\text{to first order}). \quad (8)$$

- (b) We go up in the truncation of all the quantities $\delta\mathcal{A}$ to the second order in $\delta\varphi_i$ and we study the sign of δS . Using the results of the previous step and recalling (7), we know that the first-order contribution vanishes, so that it is always possible to rewrite $E := -\delta S$ as a quadratic functional in $\delta\varphi_i$. The Gibbs stability criterion requires us to impose its positive definiteness.

¹ We avoid notations like ‘ δ ’ and ‘ δ^2 ’ for first and second order variations. We, instead, introduce $\delta\mathcal{A}$ as an exact difference, which is later approximated according to the need.

² If we interpret $\varphi_i + \delta\varphi_i$ as the state just after an external kick has been impressed, the constraint (7) implies that φ_i should not be interpreted as the state *before* the kick (because the kick might not conserve U). Rather, φ_i is the *end* state of the evolution of $\varphi_i + \delta\varphi_i$, which is reached once thermodynamic equilibrium is re-established.

Now, since in equilibrium the entropy is conserved (it cannot increase further once it is maximal), the inequality (3) implies that E cannot increase with time (namely, $E[\Sigma'] \leq E[\Sigma]$ for Σ' future of Σ). This, combined with the requirement that $E > 0$ whenever $\delta\varphi_i \neq 0$, is a sufficient condition of Lyapunov stability (more precisely, perturbations have a bounded square integral norm [42, 43]).

Furthermore, Hiscock and Lindblom proved (see proposition B of the appendix of [42]), with an argument that can be applied to every theory consistent with (1), that $E > 0$ is also a necessary condition of stability. The intuitive idea is that, for $E = -\delta S$ to approach a finite value for large times, $\nabla_a \delta s^a$ must converge to zero. Thus, all solutions of stable dissipative theories asymptotically converge to solutions of non-dissipative theories. However, the physical properties of non-dissipative theories are determined by the equilibrium equation of state, which (if computed from statistical mechanics) gives positive $-\delta S$ by construction. As $E[\text{initial}] \geq E[\text{final}]$, the positive definiteness of $E = -\delta S$ follows.

Before moving to the concrete examples, let us make a further comment about the constraints (7). In all the examples that follow (both in the main text and in the supplementary material (<https://stacks.iop.org/CQG/38/21LT02/mmedia>)), we deal with all constraints in an exact way, in the sense that (7) is only used to perform exact cancellations, which would remain valid also at higher orders than the second. However, often one may find it more convenient to work with unconstrained variations. In supplementary material: part 1, we show how to convert the maximum entropy principle at fixed energy and particle number into the minimum grand-potential principle, with completely free variations. This can, sometimes, make calculations easier (the final result is, of course, the same).

3. Examples

To illustrate how the method works in practice, we consider the simplest possible causal theory for dissipation: the divergence-type theory [5]. Adopting the notation of Geroch and Lindblom [43], the theory is built using three tensor fields, $\zeta_A = (\zeta, \zeta_a, \zeta_{ab})$, and postulates that there is a generating function $\chi = \chi(\zeta_A)$ such that

$$N^{aA} = \frac{\partial^2 \chi}{\partial \zeta_a \partial \zeta_A}, \quad (9)$$

where we have grouped the three fluxes of the theory using the notation $N^{aA} = (N^a, T^{ab}, A^{abc})$. The entropy current is given by the formula

$$s^a = \frac{\partial \chi}{\partial \zeta_a} - \zeta_A N^{aA}. \quad (10)$$

We compare the two states ζ_A and $\zeta_A + \delta\zeta_A$ and consider the second-order variation of the entropy current:

$$\delta s^a := s^a[\zeta_A + \delta\zeta_A] - s^a[\zeta_A] = \frac{1}{2} \frac{\partial^3 \chi}{\partial \zeta_a \partial \zeta_A \partial \zeta_B} \delta\zeta_A \delta\zeta_B - \zeta_A \delta N^{aA} - \delta\zeta_A \delta N^{aA}. \quad (11)$$

Imposing the consistency of this expression with the covariant Gibbs relation (as demanded by step (a)) produces the equilibrium conditions $\zeta = \alpha$, $\zeta_a = \beta K_a$ (with $\alpha, \beta = \text{const}$) and $\zeta_{ab} = 0$, in agreement with [43]. Combining this result with the constraint (7), we find that the term $-\zeta_A \delta N^{aA}$ in (11) does not contribute to the total flux (2), so we will use the shorthand notation

$$-\zeta_A \delta N^{aA} = (\text{zfc}), \quad (12)$$

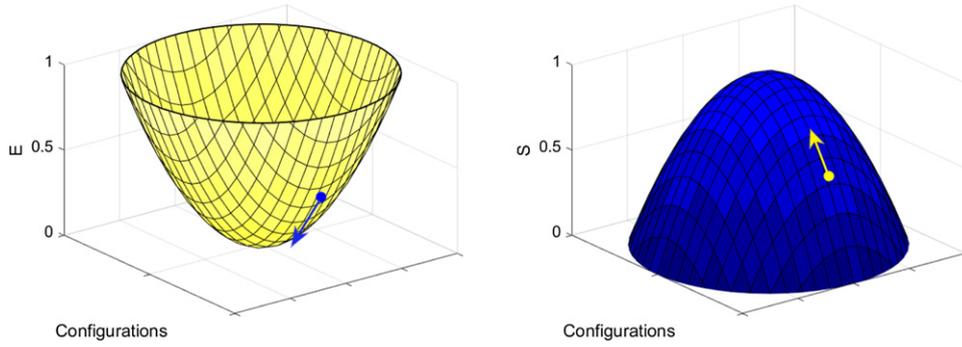


Figure 1. Two complementary views of the stability problem. Hydrodynamic view (left panel): the perturbation has a sort of ‘energy functional’ $E \geq 0$, which decreases with time and should eventually converge to zero. Thermodynamic view (right panel): the perturbation reduces the entropy S ; dissipation makes S grow again to reach S_{eq} . The two pictures are connected by the equation $E = S_{\text{eq}} - S$. The horizontal axes represent the abstract configuration space of the fluid, where each point is a global choice of $\delta\varphi_i$. The maximum of the entropy ($E = 0$) is the equilibrium state ($\delta\varphi_i = 0$).

which stands for ‘zero flux contribution’. The final step consists of using (9) to write the last term in (11) explicitly, so that

$$-\delta\zeta_A \delta N^{aA} = -\frac{\partial^3 \chi}{\partial \zeta_a \partial \zeta_A \partial \zeta_B} \delta\zeta_A \delta\zeta_B, \quad (13)$$

and we finally obtain

$$\delta s^a = -\frac{1}{2} \frac{\partial^3 \chi}{\partial \zeta_a \partial \zeta_A \partial \zeta_B} \delta\zeta_A \delta\zeta_B + (\text{zfc}) = -E^a + (\text{zfc}). \quad (14)$$

The four-vector E^a is the ‘energy current’ introduced by Geroch and Lindblom [43] in equation (51), but we see here that it is actually a second-order entropy current, whose flux is the difference between the entropy in equilibrium (the state defined by ζ_A) and the entropy in the perturbed state (the state defined by $\zeta_A + \delta\zeta_A$):

$$E = -\int_{\Sigma} E^a d\Sigma_a = S_{\text{eq}} - S. \quad (15)$$

Therefore, the condition of maximality of the entropy in equilibrium ($S_{\text{eq}} \geq S$) is equivalent to the positivity requirement for the ‘energy functional’, $E \geq 0$, see figure 1. Note that most of the mathematical properties of the field equations (e.g. their symmetric-hyperbolicity) are irrelevant for this stability criterion, because our method is based on the constitutive relations (4). In this sense, this is a condition of *thermodynamic* stability, which needs to hold independently from the dynamical equations we choose.

We have applied this same method also to the Israel–Stewart theory [4], obtaining an analogous result ($\delta s^a = (\text{zfc}) - E^a$), where in this case the ‘energy current’ coincides with the one introduced by Hiscock and Lindblom [42] in their stability analysis (see supplementary material: part 2). This clarifies the physical meaning of the stability conditions they obtained, showing how one can elegantly derive the energy functional E from thermodynamic

principles only.³ Furthermore, since the theory of Eckart is a particular Israel–Stewart theory (with $\alpha_j = \beta_j = 0$ [42]) for which E fails to be positive definite, we have a direct proof that the Eckart theory is unstable because the entropy is not maximised in equilibrium (in agreement with [44]). An analogous argument applies to Landau–Lifshitz and, more in general, to any Fick-type diffusion law.

It is interesting to analyse an example which goes beyond the standard models of causal heat conduction and viscosity, like the case of a mixture of two chemical components (say, p and n) which undergo a chemical reaction



This is an instructive case of study because, as we are going to see, our method treats the conditions of hydrodynamic, thermal, diffusive and chemical stability on the same footing.

If we do not model explicitly viscous effects and relative flows, as in [17, 18], the fields of the theory can be chosen to be the energy, p -particle and n -particle densities (ρ , n_p , n_n), plus the fluid four-velocity u^a , which is normalised: $u^a u_a = -1$. The constitutive relations take the perfect-fluid form [45]

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} \quad N^a = (n_p + n_n)u^a \quad s^a = su^a, \quad (17)$$

where the equation of state depends on both particle densities,

$$ds = \frac{1}{T} d\rho - \frac{\mu_p}{T} dn_p - \frac{\mu_n}{T} dn_n, \quad (18)$$

and the pressure can be computed from the Euler relation

$$\rho + p = Ts + \mu_p n_p + \mu_n n_n. \quad (19)$$

The conserved particle current N^a given in (17) is preserved by the chemical reaction (16), which, on the other hand, does not conserve the currents $n_p u^a$ and $n_n u^a$ separately. Therefore, although we have two chemical species, they give rise to a single (not two) conserved charge N , to be held constant in the variation.

The computation is analogous to the previous case, with the caveat that the condition $u^a u_a = -1$ needs to be respected by the variation, producing the (exact) identity $2u_a \delta u^a = -\delta u^a \delta u_a$. Taking the first-order variations and imposing the stationarity condition for S produces the well-known equilibrium conditions $\mu_p/T = \text{const}$, $u^a/T = \beta K^a$ (with β constant) and $\mu_p = \mu_n$. We focus, here, on the second-order variations, a calculation that is facilitated if one starts directly from the equilibrium state. In fact, the condition that the perturbation should preserve the energy U takes the simple form $\delta T^{ab} u_b/T = (\text{zfc})$, which, employing the constitutive relations (17), can be used to prove the relation

$$-\frac{\delta(\rho u^a)}{T} = \frac{T^{ab} \delta u_b}{T} + \frac{\delta T^{ab} \delta u_b}{T} + (\text{zfc}). \quad (20)$$

The second-order variation of the entropy current is

$$\delta s^a = \delta s u^a + s \delta u^a + \delta s \delta u^a, \quad (21)$$

³ About their energy current, Hiscock and Lindblom [42] wrote: ‘there was, unfortunately, no elegant derivation which one might hope to apply to other situations. Our ‘derivation’ was in fact based on a series of modifications and generalizations of previously existing results for similar physical situations’.

with

$$\delta s = \frac{\delta \rho}{T} - \frac{\mu_p}{T}(\delta n_p + \delta n_n) + \frac{1}{2} s^{AB} \delta n_A \delta n_B, \tag{22}$$

where we have grouped the densities using the notation $n_A = (\rho, n_p, n_n)$. s^{AB} are the components of the Hessian matrix of $s(n_A)$. After a bit of manipulation, combining together the above results and imposing $\delta N^a = (\text{zfc})$, we can write the second-order correction to the entropy current in terms of a quadratic ‘energy current’, $\delta s^a = -E^a + (\text{zfc})$, with

$$E^a = \frac{\delta T_b^a \delta u^b}{T} - \frac{u^a}{2} \frac{\rho + p}{T} \delta u^b \delta u_b - \frac{u^a}{2} s^{AB} \delta n_A \delta n_B. \tag{23}$$

Following the same procedure of Hiscock and Lindblom [42], one can show that imposing $E > 0$ for any $\delta \varphi_i \neq 0$ is equivalent to requiring

$$e := T \frac{E^a n_a}{u^b n_b} = \frac{\rho + p}{2} \delta u^b \delta u_b - \frac{T}{2} s^{AB} \delta n_A \delta n_B - \delta p \lambda_a \delta u^a > 0, \tag{24}$$

where $n^a = -n^b u_b (u^a + \lambda^a)$ is the (time-like future-directed) unit normal vector to Σ and λ^a is a space-like deviation vector whose norm lies in the range $[0, 1)$. The inequality (24) produces a number of stability conditions of various kinds. Among them we recognise the standard conditions of hydrodynamic stability such as $\rho + p > 0$ and the conditions for thermal and diffusive stability [40], like

$$\begin{aligned} 0 > s^{\rho\rho} &= \left. \frac{\partial^2 s}{\partial \rho^2} \right|_{n_p, n_n} = -\frac{1}{T^2 c_v} \\ 0 > s^{pp/nm} &= -\left. \frac{\partial}{\partial n_{p/n}} \left(\frac{\mu_{p/n}}{T} \right) \right|_{\rho, n_n/p}. \end{aligned} \tag{25}$$

We obtain also the condition of chemical stability with respect to the reaction (16), namely

$$\left. \frac{\partial^2 s}{\partial n_p^2} \right|_{\rho, n_p+n_n} = (1 \quad -1) \begin{bmatrix} s^{pp} & s^{pn} \\ s^{np} & s^{nn} \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} < 0. \tag{26}$$

But there are also some additional ‘mixed’ conditions,⁴ such as

$$-T s^{AA} \geq \frac{1}{\rho + p} \left(\left. \frac{\partial p}{\partial n_A} \right|_{n_B} \right)^2 \quad \forall A = \{ \rho, p, n \} \quad (A \text{ fixed}), \tag{27}$$

which cannot be derived within standard thermodynamics, nor from the perfect-fluid limit of the hydrodynamic model, but are hydro-diffusive conditions, specific of a two-component relativistic fluid. Note that, while the standard conditions of thermal and diffusive stability are necessary to guarantee that $-E^a u_a \geq 0$, the ‘mixed’ conditions (and all the conditions that are obtained taking $\lambda^a \lambda_a = 1^-$) force E^a to be time-like future-directed [46]. For this reason, (27) is a stronger condition than (25).

We also note that the existence of the reaction (16) leads to the condition $\mu_p = \mu_n$, but it plays no direct role in the stability criterion. This implies that, if there were no reaction, but still

⁴ To obtain (27), one needs to take the limit $\lambda^a \lambda_a \rightarrow 1$ and consider perturbations of the form $\delta u^a \propto \lambda^a$, $\delta n_C = \delta_C^A \delta n_A$ for a given A .

$\mu_p = \mu_n$ was true, we would obtain exactly the same stability conditions, but the inequality (26) would be a condition of diffusive stability. We have, thus, rediscovered the Duhem–Jougeut theorem, according to which a system that is stable to diffusion is also stable to chemical reactions [40]. This is a consequence of the fact that the hydrodynamic equations (i.e. which process modifies the densities n_p and n_n) are irrelevant, but only the constitutive relations (i.e. how the change of n_p and n_n affects the entropy) matter.

There is a clear similarity between (23) and the energy current of Israel–Stewart (see supplementary material: part 2). Indeed, the procedure that leads to both is the same and the presence of two (or more) chemical species has essentially no practical consequence on the derivation. This implies that hypothetical extensions of Israel–Stewart to mixtures should not constitute a challenge for the computation of E^a . This is an important advance on conventional methods, where all the details of the hydrodynamic equations (including possible visco-chemical couplings) would need to be *explicitly* accounted for.

Our method can also be applied to theories that are structurally different. If, for example, we consider a mixture of species that do not comove with each other, the structure (17) breaks down, because a notion of fluid velocity u^a does not exist out of equilibrium (there are, instead, two distinct velocities, u_p^a and u_n^a , of respectively p -particles and n -particles). The natural formalism for describing these fluids has been formulated by Carter [8].

We have computed the ‘energy current’ E^a of Carter’s theory in the absence of superfluidity and shear stresses [9], assuming an arbitrary number of currents n_X^a ,⁵ with conjugate momenta μ_a^X , possibly in the presence of chemical reactions and relative flows. We report here only the result (for the details see supplementary material: part 3),

$$TE^a = \sum_X \left[\frac{u^a}{2} \delta n_X^b \delta \mu_b^X - u^b \delta n_X^a \delta \mu_b^X \right]. \quad (28)$$

If all the currents comove also out of equilibrium, namely $\delta n_X^a = \delta(n_X u^a)$, (28) reduces to (23). However, (28) is more general, because it is valid for completely independent variations δn_X^a and can, therefore, be used to study the stability of a fluid against spontaneous formation of relative flows (i.e. perturbations of the form $\delta n_X^a = n_X \delta u_X^a$ with $u_a \delta u_X^a = 0$ and $\delta u_X^a \neq \delta u_Y^a$).

As a last remark, we mention that there are theories in which the entropy current fails to have *strictly* non-negative four-divergence, like the frame-stabilised first-order theories [37, 38, 47]. However, this is typically the result of a first-order truncation of the entropy current. The inclusion of higher order corrections eventually restores the entropy principle [48].

In conclusion, we have converted the hydrodynamic stability, usually regarded as a mathematical problem, into a branch of non-equilibrium thermodynamics. This fills an important gap between phenomenological hydrodynamic modelling and statistical mechanics, providing a microscopic insight into the stability conditions of a fluid.

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⁵The index X runs over all the chemical species of the model, plus the entropy, given by $X = s$.

Data availability statement

No new data were created or analysed in this study.

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Applying the Gibbs stability criterion to relativistic hydrodynamics: Supplementary Material

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Part 1: Using a simple thermodynamic argument, we convert the Gibbs stability criterion into the minimum grand-potential principle. This allows us to release the constraints on δN and δU . We use this result to prove the consistency of the method with kinetic theory and statistical mechanics.

Part 2: We show that the energy current of the Israel-Stewart theory, E^a , defined in equation (44) of Hiscock and Lindblom [1] is just $-\delta s^a$, apart from a term that does not contribute to the total integral E . The strategy that we follow is precisely the one outlined in the main text.

Part 3: Using the same technique, we compute the energy current E^a of Carter's theory, with an arbitrary number of currents, in the absence of superfluidity. We show that, in the particular case of a relativistic model for heat conduction, we recover the inviscid Israel-Stewart energy current.

PART 1: MINIMUM GRAND-POTENTIAL PRINCIPLE

A. The method of the bath

As explained in the main text, the Gibbs criterion demands that

$$\delta S \leq 0 \quad \text{as long as} \quad \delta N = \delta U = 0. \quad (1)$$

The presence of the constraints makes the problem of recasting $-\delta S$ into a quadratic functional harder. This is because the first-order part of δS does not vanish for all $\delta\varphi_i$, but only for those perturbations which conserve the total energy and particle number. Luckily, there is a simple solution to this problem.

Consider a fluid (with extensive variables U_F, N_F, S_F) in weak contact with an ideal heat and particle bath (with extensive variables U_H, N_H, S_H). The latter is defined as an effectively infinite reservoir of particles and energy with equation of state [2, 3]

$$S_H = \text{const} - \alpha N_H + \beta U_H \quad \text{with} \quad \beta \geq 0, \quad (2)$$

where the parameters α and β are constants. Equation (2) expresses the fact that the heat capacity (and, likewise, every extensive quantity) of the bath is effectively infinite. The assumption that the interaction is weak means that the extensive properties of the total system “fluid + bath” are the sum of those of the two parts:

$$U = U_F + U_H \quad N = N_F + N_H \quad S = S_F + S_H. \quad (3)$$

The stability criterion (1) holds for the total system “fluid + bath”, so that we have

$$\delta S_F + \delta S_H \leq 0 \quad \text{as long as} \quad \delta N_F = -\delta N_H \quad \delta U_F = -\delta U_H. \quad (4)$$

Combining (2) with (4) we obtain

$$\delta\Omega \geq 0 \quad (5)$$

with

$$\Omega = U_F - \frac{1}{\beta} S_F - \frac{\alpha}{\beta} N_F. \quad (6)$$

Equation (5) means that the equilibrium state of a thermodynamic system (in our case, a fluid) in contact with a heat and particle bath with (red-shifted) temperature $1/\beta$ and (red-shifted) chemical potential α/β is the state that minimizes the function Ω (with no constraint). This is nothing but the minimum grand-potential principle on curved space-time, which straightforwardly generalizes its analogue on flat space-time [4, 5].

Now, let us define the functional

$$E := \beta \delta\Omega = \beta \delta U_F - \delta S_F - \alpha \delta N_F. \quad (7)$$

This functional is positive definite for all possible $\delta\varphi_i$, with or without constraints. On the other hand,

$$\begin{aligned} \text{if } \delta N_F = \delta U_F = 0 & \Rightarrow E = -\delta S_F \geq 0 \\ \text{if } \delta N_F = \delta S_F = 0 & \Rightarrow E/\beta = \delta U_F \geq 0 \\ \text{if } \delta N_F = 0 & \Rightarrow E/\beta = \delta U_F - \delta S_F/\beta \geq 0, \end{aligned} \quad (8)$$

which are respectively the maximum entropy principle (S is maximum for fixed N and U), the minimum energy principle (U is minimum for fixed N and S) and the minimum free energy principle ($F = U - S/\beta$ is minimum for fixed N). The first line of (8) is particularly interesting for us: if we compute $E = \beta \delta\Omega$ in the *absence* of constraints, we obtain a functional that automatically reduces to $-\delta S$ when $\delta N = \delta U = 0$. However, contrarily to $-\delta S$, this functional E remains positive definite also when $\delta N \delta U \neq 0$. Therefore, rather than computing $-\delta S$, one can directly compute E for free variations, and impose its positive definiteness.

Indeed, the reader can verify explicitly that, in all the examples we propose (e.g. in Part 2 of this Supplementary material), the final formula for $E = -\delta S$ (written as the total flux of a current E^a) is exactly the same formula that one would obtain imposing $E = \beta \delta\Omega$ with released constraints.

B. Application 1: stability of the equilibrium in kinetic theory

It is interesting to note that the field nature of φ_i has never been used explicitly in the paper. This implies that the Gibbs criterion should remain valid also in the context of relativistic kinetic theory. In particular, if one replaces the fields $\varphi_i = \varphi_i(x)$ with the invariant distribution function $f = f(x, p)$, counting the number of particles in a small phase-space volume centered on (x, p) , all the arguments of the paper remain valid. Let us verify it explicitly. We set our units of energy in such a way that $g_s/h^3 = 1$, where g_s is the spin degeneracy of the gas and h is Planck's constant.

The entropy current, particle current and stress-energy tensor of an ideal quantum gas are (working in local inertial coordinates)

$$\{s^a, N^a, T^{ab}\} = \int \{\mathfrak{s}, f, fp^b\} p^a \frac{d^3p}{p^0}, \quad (9)$$

where $\mathfrak{s} = \mathfrak{s}(f)$ is a function that depends on the type of particle. Following Wu [6], a reasonably general formula for \mathfrak{s} is

$$\mathfrak{s} = -f \ln f - (1 - af) \ln(1 - af) + (1 + \tilde{a}f) \ln(1 + \tilde{a}f), \quad (10)$$

where $\tilde{a} = 1 - a$. Bosons have $a = 0$, while Fermions have $a = 1$. Intermediate cases (namely $0 < a < 1$) are anyons, which can exist in 2+1 dimensions. The Maxwell-Boltzmann case is recovered taking the limit of small f , with arbitrary a . All thermodynamic equilibrium states can be computed imposing the covariant Gibbs relation $\delta s^a = -\alpha \delta N^a - \beta K_b \delta T^{ab}$ (to first order), which implies

$$\frac{d\mathfrak{s}}{df} = -\alpha - \beta_b p^b \quad \beta_b = \beta K_b. \quad (11)$$

Computing explicitly the derivative of (10) we obtain the equilibrium condition

$$(1 - af)^a (1 + \tilde{a}f)^{\tilde{a}} = f e^{-\alpha - \beta_b p^b}, \quad (12)$$

which is the covariant generalization of the equilibrium occupation law given in [6].

We compute the functional E from equation (7). It can be written as the flux of the current $E^a = -\delta s^a - \alpha \delta N^a - \beta_b \delta T^{ab}$ which, truncated to second order, reads explicitly

$$E^a = - \int \frac{d^2\mathfrak{s}}{df^2} \frac{\delta f^2}{2} p^a \frac{d^3p}{p^0}. \quad (13)$$

The first-order part in δf cancels, due to the covariant Gibbs relation. The explicit formula of the second derivative of \mathfrak{s} is

$$\frac{d^2\mathfrak{s}}{df^2} = - \frac{1}{f(1 - af)(1 + \tilde{a}f)} < 0. \quad (14)$$

The inequality follows from the fact that $0 \leq f \leq 1/a$ [6] and implies that E^a is always time-like future-directed. Recalling that Σ is always taken space-like, we get

$$E = - \int_{\Sigma} E^a d\Sigma_a \geq 0 \quad \forall \delta f. \quad (15)$$

This proves that, in kinetic theory, all thermodynamic equilibria (both rotating and non-rotating¹) in curved space-time are maximum entropy (and minimum grand-potential) states, for all types of particles. The quantity E plays the role of a bounded square-integral norm, which is always larger than 0 (whenever $\delta f \neq 0$), and can only decrease in time. In conclusion, all thermodynamic equilibria are Lyapunov-stable, as long as the H-theorem (namely $\nabla_a s^a \geq 0$ [8]) holds.

¹ To have a rotating equilibrium, it is sufficient to require that β_b is not hypersurface-orthogonal, namely $\beta_{[a} \nabla_b \beta_{c]} \neq 0$ [7].

C. Application 2: grand-canonical ensemble in curved space-times

We can use equation (7) to derive the formula for the equilibrium density operator of a fluid in curved space-time from thermodynamic principles.

Let us begin by considering a well-known inequality: given two density operators $\hat{\rho}$ and $\hat{\sigma}$, it is always true that (9)

$$\text{Tr}(\hat{\sigma} \ln \hat{\sigma}) \geq \text{Tr}(\hat{\sigma} \ln \hat{\rho}). \quad (16)$$

If we keep $\hat{\sigma}$ arbitrary and we choose $\hat{\rho}$ to be equal to

$$\hat{\rho} = \frac{e^{\alpha \hat{N} - \beta \hat{U}}}{\text{Tr} e^{\alpha \hat{N} - \beta \hat{U}}}, \quad (17)$$

where \hat{U} and \hat{N} are the quantum energy and particle operators, then equation (16) becomes

$$\Omega[\hat{\sigma}] \geq \Omega[\hat{\rho}] \quad \forall \hat{\sigma}, \quad (18)$$

where

$$\Omega[\hat{\sigma}] = \text{Tr}(\hat{\sigma} \hat{U}) + \frac{\text{Tr}(\hat{\sigma} \ln \hat{\sigma})}{\beta} - \frac{\alpha}{\beta} \text{Tr}(\hat{\sigma} \hat{N}). \quad (19)$$

Considering that the bridge between thermodynamics and statistical mechanics is built by making the identifications

$$\{U, N, S\} = \text{Tr}(\hat{\sigma} \{ \hat{U}, \hat{N}, -\ln \hat{\sigma} \}), \quad (20)$$

it follows that $\Omega[\hat{\sigma}]$ coincides with the functional Ω introduced in equation (6). Therefore, recalling that in equation (18) $\hat{\sigma}$ is completely arbitrary, we can conclude, invoking the minimum grand-potential principle, that (17) is the equilibrium density operator.

We can rewrite equation (17) in a slightly more familiar form. Recalling that the partition function Z and the inverse-temperature four-vector β^b are given by

$$Z = \text{Tr} e^{\alpha \hat{N} - \beta \hat{U}} \quad \beta^b = \beta K^b, \quad (21)$$

and considering that the operators \hat{U} and \hat{N} can be written as the fluxes

$$\{ \hat{U}, \hat{N} \} = \int_{\Sigma} \{ K_b \hat{T}^{ab}, -\hat{N}^a \} d\Sigma_a, \quad (22)$$

equation (17) becomes

$$\hat{\rho} = \frac{1}{Z} \exp \int_{\Sigma} \left(-\alpha \hat{N}^a - \beta_b \hat{T}^{ab} \right) d\Sigma_a. \quad (23)$$

This is the well-known formula for the equilibrium density operator of relativistic fluids in curved space-time (10-13).

The present discussion shows that the Gibbs stability criterion, as it is formulated in the main text, is fully consistent with Zubarev's approach to relativistic statistical mechanics. This remains true also in the fully non-linear regime, considering that, for the argument above to be valid, $\hat{\sigma} - \hat{\rho}$ does not need to be small.

PART 2: ISRAEL-STEWART THEORY

D. Notation

We recall that the signature is $(-, +, +, +)$ and $c = k_B = 1$. We adopt exactly the same notation as Hiscock and Lindblom [1], with only three differences: for us s is the entropy per unit volume, σ is the entropy per particle ($s = n\sigma$) and the symbol Θ of [1] is replaced by the more conventional notation μ/T . This is done to guarantee coherence of notation with the main text.

E. The constitutive relations of the Israel-Stewart theory

We interpret the Israel-Stewart theory as a field theory for the tensor fields

$$(\varphi_i) = (u^a, \rho, n, \tau, q^a, \tau^{ab}), \quad (24)$$

representing respectively the flow velocity, the rest-frame energy and particle densities, the bulk-viscous stress, the heat flux and the shear-viscous stress. They satisfy the algebraic constraints

$$u^a u_a + 1 = u^a q_a = u^a \tau_{ab} = \tau_{[ab]} = \tau_a^a = 0. \quad (25)$$

Introducing the projector $q^{ab} = g^{ab} + u^a u^b$, the constitutive relations for the conserved fluxes are

$$\begin{aligned} T^{ab} &= \rho u^a u^b + (p + \tau) q^{ab} + u^a q^b + u^b q^a + \tau^{ab} \\ N^a &= n u^a \end{aligned} \quad (26)$$

and the one for the entropy current is

$$s^a = s u^a + \frac{q^a}{T} - (\beta_0 \tau^2 + \beta_1 q^b q_b + \beta_2 \tau_{bc} \tau^{bc}) \frac{u^a}{2T} + \frac{\alpha_0 \tau q^a}{T} + \frac{\alpha_1 \tau_b^a q^b}{T}, \quad (27)$$

where α_j and β_j are some expansion coefficients. The quantities s and p (representing the equilibrium entropy density and pressure) are connected to ρ and n by means of the equilibrium equation of state $s = s(\rho, n)$, hence (defined the equilibrium temperature T and chemical potential μ) we have

$$ds = \frac{1}{T} d\rho - \frac{\mu}{T} dn \quad (28)$$

and

$$\rho + p = Ts + \mu n. \quad (29)$$

F. The equilibrium states

The fact that the equilibrium states of the Israel-Stewart theory can be computed from an entropy principle is a well-known foundational feature of the theory [14]. Therefore, we will not perform the step (i) of the method (namely the first-order analysis) explicitly, as we already know that the equilibrium conditions that we would obtain from the requirement $\delta S = 0$ (at the first order) are precisely the conditions of zero entropy production ($\nabla_a s^a = 0$) found by Hiscock and Lindblom [1], namely:

$$\tau = q^a = \tau^{ab} = 0 \quad (30)$$

and

$$\frac{\mu}{T} = \alpha \quad \frac{u^a}{T} = \beta K^a \quad \alpha, \beta = \text{const} \quad \beta > 0. \quad (31)$$

We recall that the physical setting we are considering is the one outlined in our letter: stationary background spacetime, with a unique time-like future-directed symmetry generator K^a .

G. Constraints on the second-order variations

The whole study is based on the comparison between an equilibrium state φ_i , which obeys the conditions (30)-(31), and a slightly perturbed state $\varphi_i + \delta\varphi_i$, which models a small deviation from equilibrium. Both these states are assumed to obey the Israel-Stewart hydrodynamic equations (equations that, however, we do not need to introduce explicitly). The variations $\delta\varphi_i$ need to obey some constraints. First of all, since the algebraic constraints (25) must hold for both φ_i and $\varphi_i + \delta\varphi_i$, this produces the following exact identities:

$$u^a \delta u_a = -\frac{\delta u^a \delta u_a}{2} \quad u^a \delta q_a = -\delta u^a \delta q_a \quad u^a \delta \tau_{ab} = -\delta u^a \delta \tau_{ab} \quad \delta \tau_{[ab]} = \delta \tau_a^a = 0, \quad (32)$$

where we made use also of the condition (30), to be imposed on the unperturbed fields. We, furthermore, recall that the metric tensor is treated as a fixed background field, which is unaffected by the perturbation (implying that, e.g., $\delta u_a = g_{ab} \delta u^b$). The identities (32) are very useful, because they can convert quantities which look to be of first order in the perturbation (such as $u^a \delta q_a$), into quantities that are manifestly quadratic in the variations (in our example, $-\delta u^a \delta q_a$).

The other crucial constraints come from the requirement that $\delta N = \delta U = 0$. More explicitly, we need to impose

$$\{ \delta N, \delta U \} = \int_{\Sigma} \{ -\delta N^a, \delta T^{ab} K_b \} d\Sigma_a = 0. \quad (33)$$

Recalling (31) and adopting the same notation as in the main text, we can rewrite the aforementioned constraints in the following simpler forms:

$$\delta N^a = (\text{zfc}) \quad \delta T^{ab} \frac{u_b}{T} = (\text{zfc}). \quad (34)$$

The first equation can be immediately converted into a constraint on the fields n and u^a :

$$\delta n u^a + n \delta u^a + \delta n \delta u^a = (\text{zfc}). \quad (35)$$

Furthermore, the second equation of (34) can be rewritten² in the more useful form

$$\frac{\delta \rho u^a}{T} + \frac{\rho \delta u^a}{T} + \frac{\delta \rho \delta u^a}{T} + \frac{\delta q^a}{T} + \frac{T^{ab} \delta u_b}{T} + \frac{\delta T^{ab} \delta u_b}{T} = (\text{zfc}). \quad (36)$$

H. Perturbation to the entropy current

We only need to make the second-order expansion of the constitutive relation (27) in terms of $\delta\varphi_i$, where we recall that the selection of fields φ_i to be used as free variables is made in (24). The calculation is straightforward:

$$\begin{aligned} \delta s^a = & \left(\frac{\delta \rho}{T} - \frac{\mu}{T} \delta n + \frac{1}{2} s^{AB} \delta n_A \delta n_B \right) u^a + s \delta u^a + (\delta \rho - \mu \delta n) \frac{\delta u^a}{T} + \frac{\delta q^a}{T} \\ & - \frac{\delta q^a \delta T}{T^2} - (\beta_0 \delta \tau \delta \tau + \beta_1 \delta q^b \delta q_b + \beta_2 \delta \tau_{bc} \delta \tau^{bc}) \frac{u^a}{2T} + \frac{\alpha_0 \delta \tau \delta q^a}{T} + \frac{\alpha_1 \delta \tau_b^a \delta q^b}{T}, \end{aligned} \quad (37)$$

where we introduced the compact notation $n_A = (\rho, n)$ and s^{AB} is the Hessian matrix of $s(n_A)$. We can use the constraints (35) and (36), together with the first equilibrium condition of (31) to rewrite the first line of (37) in a more convenient form:

$$\begin{aligned} \delta s^a = & (\text{zfc}) + \frac{u^a}{2} s^{AB} \delta n_A \delta n_B + (Ts + \mu n - \rho) \frac{\delta u^a}{T} - \frac{T^{ab} \delta u_b}{T} - \frac{\delta T^{ab} \delta u_b}{T} \\ & - \frac{\delta q^a \delta T}{T^2} - (\beta_0 \delta \tau \delta \tau + \beta_1 \delta q^b \delta q_b + \beta_2 \delta \tau_{bc} \delta \tau^{bc}) \frac{u^a}{2T} + \frac{\alpha_0 \delta \tau \delta q^a}{T} + \frac{\alpha_1 \delta \tau_b^a \delta q^b}{T}. \end{aligned} \quad (38)$$

² Start from the general identity $\rho u^a + q^a + T^{ab} u_b = 0$, valid on both the equilibrium and the perturbed state.

However, recalling the Euler relation (29), it is easy to show that

$$(Ts + \mu n - \rho) \frac{\delta u^a}{T} - \frac{T^{ab} \delta u_b}{T} = (\rho + p) \frac{u^a}{T} \frac{\delta u^b \delta u_b}{2}, \quad (39)$$

which can be inserted into (38), giving

$$\delta s^a = (\text{zfc}) - E^a, \quad (40)$$

with

$$\begin{aligned} TE^a &= \delta T_b^a \delta u^b - \frac{1}{2}(\rho + p) u^a \delta u^b \delta u_b - \frac{u^a}{2} T s^{AB} \delta n_A \delta n_B \\ &\quad + \frac{\delta q^a \delta T}{T} + (\beta_0 \delta \tau \delta \tau + \beta_1 \delta q^b \delta q_b + \beta_2 \delta \tau_{bc} \delta \tau^{bc}) \frac{u^a}{2} - \alpha_0 \delta \tau \delta q^a - \alpha_1 \delta \tau_b^a \delta q^b, \end{aligned} \quad (41)$$

which constitutes the quadratic ‘‘energy current’’ we were looking for.

I. Comparison with the energy current of Hiscock and Lindblom

Note that, if our task was just to compute the energy current E^a of Israel-Stewart, we could just stop here. In fact, we have already obtained a formula for it: equation (41). However, if we compare it with equation (44) of Hiscock and Lindblom [1],

$$\begin{aligned} TE^a &= \delta T_b^a \delta u^b - \frac{1}{2}(\rho + p) u^a \delta u^b \delta u_b + \frac{1}{\rho + p} \left(\frac{\partial \rho}{\partial p} \Big|_{\sigma} (\delta p)^2 + \frac{\partial \rho}{\partial \sigma} \Big|_p \frac{\partial p}{\partial \sigma} \Big|_{\mu/T} (\delta \sigma)^2 \right) \frac{u^a}{2} \\ &\quad + \frac{\delta q^a \delta T}{T} + (\beta_0 \delta \tau \delta \tau + \beta_1 \delta q^b \delta q_b + \beta_2 \delta \tau_{bc} \delta \tau^{bc}) \frac{u^a}{2} - \alpha_0 \delta \tau \delta q^a - \alpha_1 \delta \tau_b^a \delta q^b, \end{aligned} \quad (42)$$

we see that the two energy currents in (41) and (42) are the same only if one manages to show that

$$-T s^{AB} \delta n_A \delta n_B = \frac{1}{\rho + p} \left(\frac{\partial \rho}{\partial p} \Big|_{\sigma} (\delta p)^2 + \frac{\partial \rho}{\partial \sigma} \Big|_p \frac{\partial p}{\partial \sigma} \Big|_{\mu/T} (\delta \sigma)^2 \right). \quad (43)$$

It turns out that this identity is, indeed, true, proving that our energy current is exactly the same as the one of Hiscock and Lindblom [1] and confirming the argument of Gavassino *et al.* [15], according to which the stability conditions of Israel-Stewart are precisely those conditions for which the entropy is maximal in equilibrium. However, proving (43) is not so straightforward, and requires some elaborate thermodynamic manipulations, which are presented below.

First of all, we list the thermodynamic identities that are needed to prove (43):

$$\frac{dT}{T} = \frac{dp}{\rho + p} - \frac{nT}{\rho + p} d\left(\frac{\mu}{T}\right), \quad (44)$$

$$\frac{dn}{n} = \frac{dp}{\rho + p} - \frac{Tn}{\rho + p} d\left(\frac{\mu}{T}\right), \quad (45)$$

$$\frac{\partial \rho}{\partial \sigma} \Big|_p = n^2 T^2 \frac{\partial}{\partial p} \left(\frac{\mu}{T} \right) \Big|_{\sigma}, \quad (46)$$

$$\frac{\partial \rho}{\partial \sigma} \Big|_p \frac{\partial p}{\partial \sigma} \Big|_{\mu/T} = -n^2 T^2 \frac{\partial}{\partial \sigma} \left(\frac{\mu}{T} \right) \Big|_p. \quad (47)$$

Equations (44) and (45) can be straightforwardly derived from the differentials $dp = s dT + n d\mu$ and (28). Equation (46) and (47) are simply the identities (89) and (94) of Hiscock and Lindblom [1].

Our proof of the identity (43) follows four steps. First, using (28), it is easy to show that (since all the terms are quadratic in the perturbation, we can use first-order identities to make changes of variables)

$$\mathcal{Z} := -T s^{AB} \delta n_A \delta n_B = \delta n \delta \mu + \delta s \delta T = T \delta n \delta \left(\frac{\mu}{T} \right) + \frac{\delta \rho \delta T}{T}. \quad (48)$$

Secondly, we can use the identities (44) and (45) to justify the following equalities:

$$\mathcal{Z} = T \delta n \delta \left(\frac{\mu}{T} \right) + \frac{\delta \rho \delta p}{\rho + p} - \frac{nT \delta \rho}{\rho + p} \delta \left(\frac{\mu}{T} \right) = \frac{\delta \rho \delta p}{\rho + p} - \frac{n^2 T^2 \delta \sigma}{\rho + p} \delta \left(\frac{\mu}{T} \right). \quad (49)$$

The third step consists of writing $\delta \rho$ and $\delta(\mu/T)$ in terms of δp and $\delta \sigma$,

$$\delta \rho = \left. \frac{\partial \rho}{\partial p} \right|_{\sigma} \delta p + \left. \frac{\partial \rho}{\partial \sigma} \right|_p \delta \sigma \quad \delta \left(\frac{\mu}{T} \right) = \left. \frac{\partial}{\partial p} \left(\frac{\mu}{T} \right) \right|_{\sigma} \delta p + \left. \frac{\partial}{\partial \sigma} \left(\frac{\mu}{T} \right) \right|_p \delta \sigma, \quad (50)$$

so that we find

$$(\rho + p)\mathcal{Z} = \left. \frac{\partial \rho}{\partial p} \right|_{\sigma} (\delta p)^2 + \left[\left. \frac{\partial \rho}{\partial \sigma} \right|_p - n^2 T^2 \left. \frac{\partial}{\partial p} \left(\frac{\mu}{T} \right) \right|_{\sigma} \right] \delta p \delta \sigma - n^2 T^2 \left. \frac{\partial}{\partial \sigma} \left(\frac{\mu}{T} \right) \right|_p (\delta \sigma)^2. \quad (51)$$

Finally, we only need to use the identities (46) and (47) to obtain

$$(\rho + p)\mathcal{Z} = \left. \frac{\partial \rho}{\partial p} \right|_{\sigma} (\delta p)^2 + \left. \frac{\partial \rho}{\partial \sigma} \right|_p \left. \frac{\partial p}{\partial \sigma} \right|_{\mu/T} (\delta \sigma)^2, \quad (52)$$

which is what we wanted to prove.

PART 3: CARTER'S THEORY

J. Notation

We adopt exactly the same notation as Carter and Khalatnikov [16], with the only difference that their quantities Ψ , Θ_a and Θ will be denoted by p , T_a and T (p and T reduce to the ordinary pressure and temperature in equilibrium). This is done to ensure notational conformity with Part 2. Equation (28) of the main text is explicitly obtained in subsection [N].

K. The constitutive relations of Carter's theory

We choose the momentum-based representation, according to which the fundamental fields of the theory are the momenta

$$(\varphi_i) = (\mu_a^X). \quad (53)$$

The theory postulates that there is a scalar field p (representing the total pressure) such that the constitutive relations for the currents n_X^a , entropy current included (for $X = s$ we impose $n_X^a = n_s^a = s^a$), are given by the differential (at fixed metric components)

$$dp = -n_X^a d\mu_a^X, \quad (54)$$

while the constitutive relation for the stress-energy tensor is

$$T_b^a = pg_b^a + n_X^a \mu_b^X. \quad (55)$$

We are adopting Einstein's summation convention for the chemical index X , including $X = s$. The covector μ_a^s , which is associated with the entropy current s^a , is denoted by T_a . We assume that no species is superfluid, which implies that no constraint is imposed on the covector fields μ_a^X (i.e. there is no conserved winding number [2]).

L. The equilibrium states

Also in Carter's theory the equilibrium states can be easily computed from the maximum entropy principle. Since the calculation is straightforward, here we report only the result. Given the definition of the inverse-temperature four-vector

$$\beta^a := \frac{-s^a}{s^b T_b}, \quad (56)$$

one finds that all the currents are collinear to β^a in equilibrium (there is no superfluidity here [2]),

$$\frac{n_X^a}{\sqrt{-n_X^b n_{Xb}}} = \frac{\beta^a}{\sqrt{-\beta^b \beta_b}} =: u^a \quad \forall X, \quad (57)$$

so that u^a represents the *equilibrium* collective fluid velocity of all the species. In this configuration the fluid becomes indistinguishable from a multi-constituent perfect fluid, like the pn -mixture presented in the main body. In equilibrium (and only in equilibrium), $T = -s^a T_a / s$ is the ordinary temperature of the mixture and we can write

$$\beta^a = \frac{u^a}{T} \quad n_X^a = n_X u^a. \quad (58)$$

Apart from the collinearity condition, which may be seen as the condition of local thermodynamic equilibrium (analogous to [30]), we have some conditions of global equilibrium (analogous to [31]):

$$\beta^a \mu_a^X = -\alpha^X \quad \beta^a = \beta K^a \quad \alpha^X, \beta = \text{const} \quad \beta > 0. \quad (59)$$

Note that $\alpha^s = 1$, identically. Finally, coherently with our remarks on the Duhem-Jougeut theorem, one can verify that the possible presence of chemical reactions does not modify any of these equilibrium conditions, but it imposes constraints on the possible values of the constants α^X . For example, the presence of a reaction like



would produce the constraint

$$\alpha^X + 2\alpha^Y = 5\alpha^Z. \quad (61)$$

M. Constraints on the second-order variations

Contrary to the case of the Israel-Stewart theory, there is no local constraint to be imposed on the variation of the fields μ_a^X . All the constraints have a global character. The requirement that the variation should preserve the values of the integrals of motion produces constraints

$$\alpha^X \delta n_X^a = (\text{zfc}) + \delta s^a \quad \delta T_b^a \beta^b = (\text{zfc}). \quad (62)$$

While the second one is the obvious analogue of the second relation of (34), the first one requires a bit of explanation. Let Q_Y be a basis of independent conserved (i.e. unchanged by the chemical reactions) charges of the fluid,

$$Q_Y = \sum_{X \neq s} q_Y^X N_X, \quad (63)$$

where q_Y^X is a matrix of constant coefficients, measuring the amount of charge Y carried by an individual particle of type X (N_X is the total number of X -particles). All the equilibrium conditions of the kind (61) are simultaneously respected if and only if there is a set of constant coefficients λ^Y (one for every charge Q_Y) such that

$$\alpha^X = \sum_Y \lambda^Y q_Y^X \quad \forall X \neq s. \quad (64)$$

Since the perturbation conserves the values of the constants of motion of the fluid, we need to impose the exact constraint $\delta Q_Y = 0$, which implies

$$\sum_{X \neq s} \alpha^X \delta N_X = \sum_{(X \neq s), Y} \lambda^Y q_Y^X \delta N_X = \sum_Y \lambda^Y \delta Q_Y = 0, \quad (65)$$

which, written in terms of currents, becomes

$$\sum_{X \neq s} \alpha^X \delta n_X^a = (\text{zfc}). \quad (66)$$

Adding to both sides δs^a , and recalling that $\alpha^s = 1$, we finally obtain the first relation in (62).

N. Perturbation to the entropy current

We now derive equation (28) of the main text. Let us, first of all, consider the perturbation to the stress-energy tensor:

$$\delta T_b^a = \delta p g_b^a + \delta n_X^a \mu_b^X + n_X^a \delta \mu_b^X + \delta n_X^a \delta \mu_b^X. \quad (67)$$

If we contract this variation with β^b and impose the constraints (62) we find

$$\delta s^a = (\text{zfc}) + \beta^a \delta p + \beta^b n_X^a \delta \mu_b^X + \beta^b \delta n_X^a \delta \mu_b^X. \quad (68)$$

The collinearity condition (58) implies that $\beta^b n_X^a = \beta^a n_X^b$ so we obtain

$$\delta s^a = (\text{zfc}) + \beta^a (\delta p + n_X^b \delta \mu_b^X) + \beta^b \delta n_X^a \delta \mu_b^X. \quad (69)$$

Finally, the second-order variation δp can be written in the convenient form

$$\delta p = -n_X^b \delta \mu_b^X - \frac{1}{2} \delta n_X^b \delta \mu_b^X, \quad (70)$$

so that again we have $\delta s^a = (\text{zfc}) - E^a$, with

$$TE^a = \frac{u^a}{2} \delta n_X^b \delta \mu_b^X - u^b \delta n_X^a \delta \mu_b^X. \quad (71)$$

This Hiscock-Lindblom-type current can be used to derive all the stability conditions of a generic (non-superfluid) Carter's fluid, both in the presence and in the absence of chemical reactions.

O. A particular case: Carter's model for heat conduction

Carter's model for heat conduction [17] is built using only two covectors (T_a and μ_a) as fundamental fields, which are dual respectively to the entropy and the particle current (s^a and N^a). Priou [18] has shown that, close to equilibrium, this model becomes very similar to an Israel-Stewart heat-conductive (but inviscid) fluid. This comparison becomes more evident if one makes the decomposition (see Lopez-Monsalvo and Andersson [19] for all the details)

$$N^a = nu^a \quad s^a = su^a + \frac{q^a}{T} \quad \mu_a = \mu u_a + \frac{\mathcal{A} q_a}{T} \quad T_a = T u_a + \frac{\mathcal{C} q_a}{T}, \quad (72)$$

where

$$u^a q_a = 0, \quad \mathcal{C} = \beta_{IS} T^2 \quad \text{and} \quad T = \mathcal{C} s + \mathcal{A} n. \quad (73)$$

Note that, within Carter's approach, the quantities n , s , T , μ and u^a are built from the geometrical decomposition (72) directly as non-equilibrium quantities, generalizing the corresponding equilibrium fields. Contrary to the Israel-Stewart case, they are not used as identifiers of a fiducial local thermodynamic equilibrium state. In fact, n and s are not connected to T and μ by the equilibrium equation of state. The coefficient β_{IS} is a sort of Carter's analogue of the thermodynamic coefficient β_1 appearing in (27).

If we insert (72) into (71), truncating the result at the second order, we obtain

$$\begin{aligned} TE^a &= \frac{u^a}{2} (\delta n \delta \mu + \delta s \delta T) + \frac{u^a}{2} (n\mu + sT) \delta u^b \delta u_b + \delta u^a (n\delta \mu + s\delta T) \\ &+ u^a \delta q^b \delta u_b + \frac{\delta q^a \delta T}{T} + \frac{u^a}{2} \beta_{IS} \delta q^b \delta q_b. \end{aligned} \quad (74)$$

To facilitate the interpretation of this current, we can insert (72) into (54) and (55), to obtain the exact formulas

$$dp = n d\mu + s dT - \frac{q^a}{T} d\left(\frac{\mathcal{C} q_a}{T}\right) \quad (75)$$

$$\rho := T^{ab} u_a u_b = n\mu + sT - p. \quad (76)$$

$$T^{ab} = (\rho + p) u^a u^b + p g^{ab} + u^a q^b + u^b q^a + \beta_{IS} q^a q^b. \quad (77)$$

These can be easily used to show that

$$\delta T_b^a \delta u^b - \frac{u^a}{2} (\rho + p) \delta u^b \delta u_b = \frac{u^a}{2} (n\mu + sT) \delta u^b \delta u_b + \delta u^a (n\delta \mu + s\delta T) + u^a \delta q^b \delta u_b, \quad (78)$$

so that equation (74) takes the more familiar form

$$\begin{aligned} TE^a &= \delta T_b^a \delta u^b - \frac{u^a}{2} (\rho + p) \delta u^b \delta u_b + \frac{u^a}{2} (\delta n \delta \mu + \delta s \delta T) \\ &+ \frac{\delta q^a \delta T}{T} + \frac{u^a}{2} \beta_{IS} \delta q^b \delta q_b. \end{aligned} \quad (79)$$

Recalling equation (48), we see that this formula for E^a is indistinguishable from the inviscid limit ($\delta \tau = \delta \tau^{ab} = 0$) of the energy current (41) of Israel-Stewart.

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