“HIGH ENERGY ASTROPHYSICS”
(A subjective introduction)

III. Gas processes

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Story so far ...

- in high energy astrophysics energies of particles are high enough so special relativity effects cannot be neglected
    - time dilation
    - length contraction
    - relativistic beaming
    - Doppler effect

- four vectors and tensor calculus

Fluid

Collection of particles may be treated as a *fluid* if the particles collide frequently enough so the system may be described by the mean quantities (pressure, temperature, mass density instead of force, particle velocity, particle mass)

Average distance between collisions is small compared to the macroscopic size of the system.

Number of particles is large.

System may posses a mean bulk velocity, but it is not affected by the random velocities of the particles.

Collisions

- *cross section* σ

  measure of probability of interaction; in [cm2]

- *mean free path* λ

  the average distance between two collisions

COLLISIONS:

- neutral particles
- charged particles
  - strong collisions
  - weak collisions

Collisions

NEUTRAL PARTICLES

- mean free path

$$\lambda = \frac{1}{n\sigma}$$

for neutral hydrogen

$$\sigma = \pi (2 a_0)^2, \qquad a_0 = \frac{\hbar^2}{m_e c^2} = 5\times 10^{-9}\ \mathrm{cm}$$

$$\lambda \approx 3\times 10^{15} \left(\frac{n}{1\ \mathrm{cm}^{-3}}\right)^{-1} \mathrm{cm}$$

- random velocities

$$v_{\mathrm{rms}} = \left(\frac{3 k_{\mathrm{B}} T}{\mu m_{\mathrm{H}}}\right)^{1/2}, \qquad \mu = \frac{\bar{m}}{m_{\mathrm{H}}} = \frac{\sum_i n_i A_i}{\sum_i n_i + n_e}$$

$$\mu_{\mathrm{neutral}} = \left[\frac{\sum_i X_i}{A_i}\right]^{-1}, \qquad \mu_{\mathrm{ionized}} = \left[\frac{\sum_i X_i (Z_i + 1)}{A_i}\right]^{-1}$$

Collisions

CHARGED PARTICLES

- Coulomb force

$$F_{\text{Coulomb}} = \frac{q_1 q_2}{r^2}$$

- strong vs. weak collisions

$$\frac{Z e^2}{r} > \frac{1}{2} m_e v^2$$

strong collision radius r$_S$

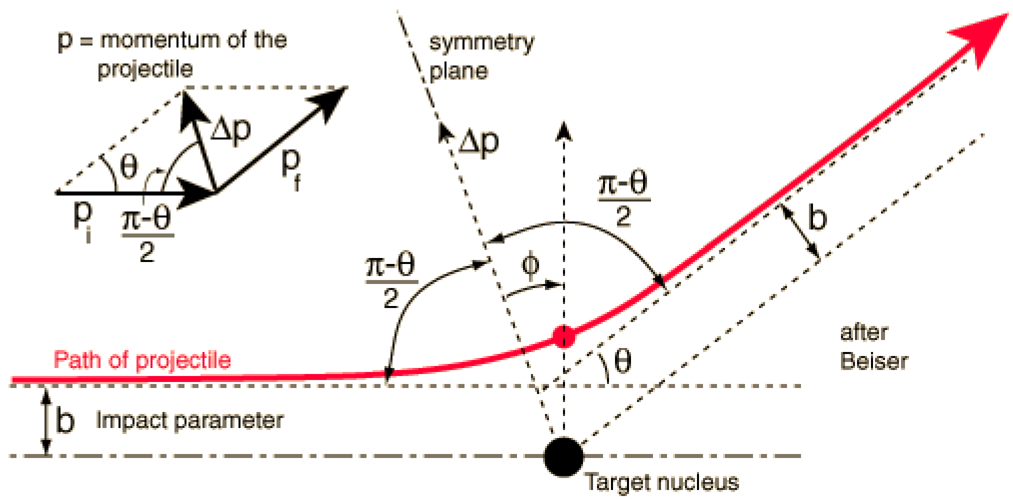
$$r < r_S = \frac{2 Z e^2}{m_e v^2}$$

timescale between collisions

$$t_{\text{strong}} = \frac{\lambda}{v} = \frac{1}{v n \sigma} = \frac{m_e v^3}{4 \pi n Z^2 e^4}, \qquad \sigma = \pi r_S^2$$

Collisions

CHARGED PARTICLES – WEAK COLLISIONS



$$\Delta p_{\perp} = \int_{-\infty}^{+\infty} F_{\perp}(t)\,dt = \frac{Ze^2}{bv}\int_{-\infty}^{+\infty}(1+s^2)^{-3/2}\,ds = \frac{Ze^2}{bv}\left|\frac{s}{(1+s^2)^{1/2}}\right|_{-\infty}^{+\infty} = \frac{2Ze^2}{bv}$$

$$\Delta v_{\perp} = \frac{\Delta p_{\perp}}{m_e} = \frac{2Ze^2}{m_e b v}$$

Collisions

CHARGED PARTICLES – WEAK COLLISIONS

$$\frac{d(\Delta v)^2}{dt} = n v \int_{b_{\min}}^{b_{\max}} \Delta v_\perp^2 \, 2\pi b \, db = \frac{8\pi n Z^2 e^4}{m_e^2 v} \ln \frac{b_{\max}}{b_{\min}}$$

- *Coulomb logarithm*

$$\ln \frac{b_{\max}}{b_{\min}} \equiv \ln \Lambda$$

- limits

$$b_{\min} > r_S = \frac{2 Z e^2}{m_e v^2} \quad \text{or} \quad b_{\min} > \frac{\hbar}{m_e v} \quad (\Delta p \Delta x \sim \hbar)$$

Debye length

$$b_{\max} = \left(\frac{k_B T}{4\pi n e^2} \right)^{1/2}$$

- usually ln Λ = 10-30

Collisions

CHARGED PARTICLES

- timescale

$$t_{\text{weak}} = \frac{v^2}{\frac{d(\Delta v)^2}{dt}} = \frac{m_e^2 v^3}{8\pi n Z^2 e^4 \ln\Lambda}$$

$$t_{\text{weak}} = \frac{t_{\text{strong}}}{2\ln\Lambda}$$

- distance

$$\lambda_{\text{Coulomb}} = v\, t_{\text{weak}} \approx 1.4\times 10^4 \frac{1}{Z^2 \ln\Lambda}\left(\frac{t}{1\,\text{K}}\right)^2 \left(\frac{n}{1\,\text{cm}^{-3}}\right)^{-1} \text{cm}$$

Collisions

CHARGED PARTICLES

- with magnetic field

$$\vec{F}_{\mathrm{Lorentz}} = \frac{q}{c}(\vec{v} \times \vec{B})$$

- gyroradius (Larmour radius)

$$\frac{q\,v\,B}{c} = \frac{m\,v^2}{r_{\mathrm{L}}} \quad \Rightarrow \quad r_{\mathrm{L}} = \frac{m\,v\,c}{q\,B}$$

- *Larmor frequency*

$$\omega_{\mathrm{L}} = \frac{v}{r_{\mathrm{L}}} = \frac{q\,B}{m\,c}$$

- frozen-in flow

Ionization

- ionization states

  Fe I (Fe), Fe II, Fe III, …, Fe XXVII (Fe XXVI)

- *Saha equation*

$$\frac{n_{i+1} n_e}{n_i} = \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \frac{2 Z_{i+1}(T)}{Z_i(T)} \exp\left(\frac{-U_{ionis}}{k_B T} \right) \ ,$$

  where

  n_i, n_{i+1} – number density of ions

  $Z_i(T)$, $Z_{i+1}(T)$ – *partition functions* $Z(T) = \Sigma g_i \exp(-E_i/k_B T)$

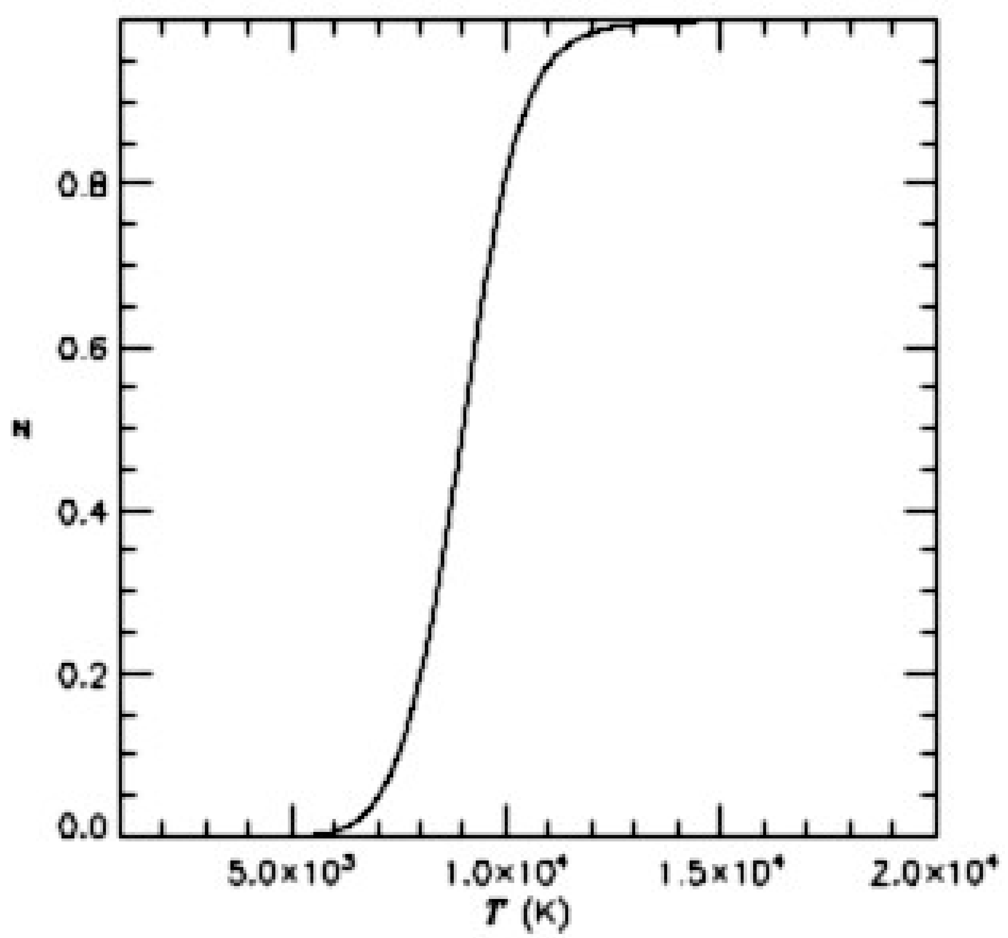
  g_i – *statistical weight* (number of degenerate states)

  U_{ionis} – *ionization potential*

- for hydrogen

$$\frac{\xi^2}{1-\xi} = 2.4 \times 10^{15} \frac{T^{3/2}}{n} \exp\left(-1.6 \times 10^5 \frac{K}{T} \right) \ , \qquad \xi \overset{def}{=} \frac{n_{HII}}{n_{HI} + n_{HII}}$$

Ionization



- *pressure ionization*

Viscosity

- transport of momentum in a shear flow

  flux of particles

$$\sim n\, v_{\rm rms}/6$$

  net flux of momentum

$$\frac{1}{6}\, n\, m\, v_{\rm rms}\left[u(y_0-\lambda)-u(y_0+\lambda)\right]$$

  viscous stress (force per unit area) and *viscosity coefficient*

$$\sigma_{\rm visc}=-\eta\frac{\partial u}{\partial y}\quad,\qquad \eta=\frac{1}{3}\, n\, m\, v_{\rm rms}\,\lambda\quad,\qquad \nu\overset{\rm def}{=}\frac{\eta}{\rho}$$

  for thermal gas

$$\eta=\frac{\left(m\, k_{\rm B}\, T\right)^{1/2}}{\sqrt{3}\,\sigma}\propto T^{1/2}$$

- *Reynolds number*

$$\mathrm{Re}=\frac{L\, v\, \rho}{\eta}$$

Equations of fluid dynamics

- Lagrangian and Eulerian derivatives

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+(\vec{v}\circ\vec{\nabla})\, f$$

- *continuity equation*

$$\frac{\partial \rho}{\partial t}=-\vec{\nabla}\circ(\rho\vec{v})$$

- *Euler equation* (momentum conservation)

$$\rho\frac{d\vec{v}}{dt}=-\vec{\nabla}P-\rho\vec{\nabla}\varphi$$

- *energy equation* (energy conservation)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho e\right)+\vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho v^2+\rho e+P\right)\vec{v}\right]=-\rho\vec{v}\cdot\vec{\nabla}\varphi+L$$

Equation of state

- *phase space distribution function* (number of particles per phase space volume $d^3x d^3p$)

$$\tilde{f} = \frac{2S+1}{h^3} f$$

$$n = \int \tilde{f}\, d^3 p \quad , \quad P = \frac{1}{3} \int p\, v\, \tilde{f}\, d^3 p \quad , \quad u = \int E\, dn$$

- ideal gas

$$f(E) = \exp\left(\frac{\mu - E}{k_B T}\right)$$

$$n = (2S+1)\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{\mu - mc^2}{k_B T}\right) \quad , \quad P = n k_B T \quad , \quad u = n m c^2 + \frac{3}{2} n k_B T$$

- electron gas, *polytropic equation* and *adiabatic exponent*

$$f(E) = \frac{1}{\exp\left(\frac{E - \mu}{k_B T}\right) + 1}$$

$$P = K \rho^\Gamma \quad , \quad \Gamma = \frac{\partial \ln P}{\partial \ln \rho} \quad , \quad \Gamma_{NR} = \frac{5}{3} \quad , \quad \Gamma_{ER} = \frac{4}{3}$$

Story so far ...

- astrophysical gas as a fluid
- collisions: neutral vs. charged (strong vs. weak)
- ionization (Saha's equation)
- viscosity
- equations of fluid dynamics and equation of state
    - sound waves in fluids

Sound waves

- small disturbance in a uniform plasma with no magnetic field and polytropic equation of state

$$v \quad , \quad P_0 + P \quad , \quad \rho_0 + \rho$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v}{\partial x} = 0 \quad , \quad \rho_0 \frac{\partial v}{\partial t} = \frac{-\partial P}{\partial x} \quad , \quad P = c_s^2 \rho \quad , \quad c_s^2 \overset{\text{def}}{=} \frac{\partial P}{\partial \rho} = \frac{\Gamma P}{\rho}$$

$$\frac{\partial^2 \rho}{\partial t^2} = c_s^2 \frac{\partial^2 \rho}{\partial x^2}$$

- *sound waves*, *Alfen waves*, and *magneto-acustic waves*

$$v_{\mathrm{A}} = \left(\frac{B^2}{4 \pi \rho} \right)^{1/2}$$

Shock waves

HYDRODYNAMIC SHOCK

- conservation of mass, momentum and energy in shock frame

$$\rho_2 v_2 = \rho_1 v_1$$

$$P_2 + \rho_2 v_2^2 = P_1 + \rho_1 v_1^2$$

$$P_2 v_2 + \left(\rho_2 e_2 + \frac{1}{2}\rho_2 v_2^2\right) v_2 = P_1 v_1 + \left(\rho_1 e_1 + \frac{1}{2}\rho_1 v_1^2\right) v_1$$

- for non-relativistic plasma

$$e = \frac{P}{(\Gamma - 1)\rho} = \frac{3P}{2\rho} \quad \Rightarrow \quad \frac{1}{2} v_2^2 + \frac{5}{2}\frac{P_2}{\rho_2} = \frac{1}{2} v_1^2 + \frac{5}{2}\frac{P_1}{\rho_1}$$

- *Rankine-Hugoniot jump conditions*

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\Gamma+1) M_1^2}{2 + (\Gamma-1) M_1^2} \quad , \quad \frac{P_2}{P_1} = \frac{2\Gamma M_1^2 - (\Gamma - 1)}{\Gamma + 1}$$

- *Mach number*

$$M^2 = \frac{v^2}{c_s^2}$$

Shock waves

HYDRODYNAMIC SHOCK

- $M_1 > 1$ – shock speed ($u=v_1$) is supersonic

- $v_2 < c_s^{\ 2}$ – in the shock frame flow is subsonic downstream and supersonic upstream

- $p_2>p_1$ and $\rho_2>\rho_1$ – the shock is compressive

$$M_1 \gg 1 \quad \Rightarrow \quad \frac{\rho_1}{\rho_2}=\frac{\Gamma-1}{\Gamma+1} \quad , \quad \frac{P_2}{P_1}=\frac{2\,\Gamma\, M_1^2}{\Gamma+1}$$

- $v_2<v_1$ and $T_2>T_1$ – the shock slows down the gas and heats it up

$$M_1 \gg 1 \quad \Rightarrow \quad \frac{v_2}{v_1}=\frac{\rho_1}{\rho_2} \quad , \quad \frac{T_2}{T_1}=\frac{\Gamma-1}{\Gamma+1}\,\frac{P_2}{P_1}$$

Shock acceleration

- energy gain in single crossing

$$v = v_2 - v_1 = \frac{3}{4} u \quad , \quad E' = E + p_x v \quad , \quad \frac{\Delta E}{E} = \frac{v}{c}$$

- particle distribution

$$\beta = 1 + \frac{v}{c} \quad , \quad E = E_0 \beta^j \quad , \quad N = N_0 P^j$$

$$\frac{\log(N/N_0)}{\log(E/E_0)} = \frac{\log P}{\log \beta} \quad \Rightarrow \quad \frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{\log P/\log\beta}$$

$$n(E)\, dE \propto E^{\log P/\log\beta - 1}\, dE = E^{-k}\, dE$$

- non-thermal distribution

Shock acceleration

- average number of particles crossing the shock nc/4
- fraction of particles advected from the shock

$$\frac{nU/4}{nc/4}=\frac{U}{c}$$

$$\log P=\log\left(1-\frac{U}{c}\right)\approx-\frac{U}{c}$$

- energy gain

$$\frac{\Delta E}{E}=\frac{v}{c}\cos\theta$$

$$\left\langle\frac{\Delta E}{E}\right\rangle=\frac{v}{c}\int_0^{\pi/2}2\cos^2\theta\sin\theta\,d\theta=\frac{2}{3}\frac{v}{c}$$

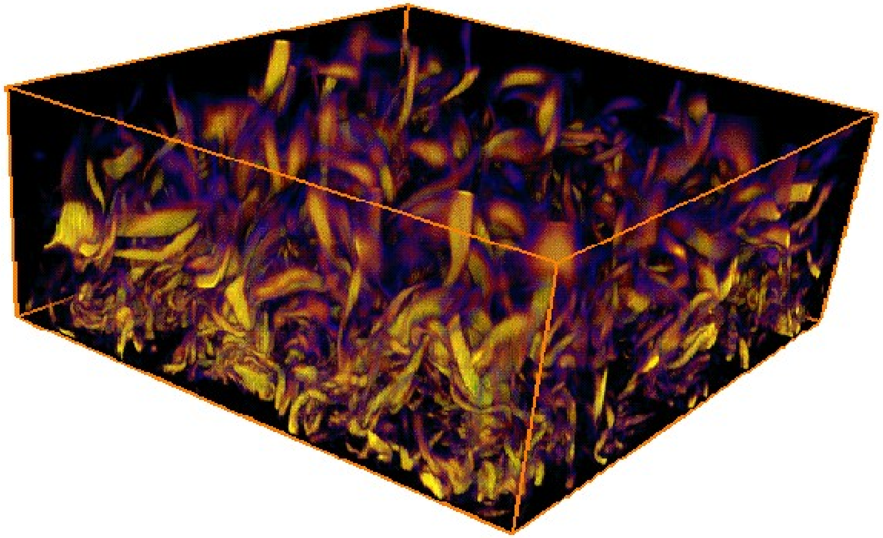
$$\beta=\frac{E}{E_0}=1+\frac{4}{3}\frac{v}{c}\quad\Rightarrow\quad\log\beta=\log\left(1+\frac{U}{c}\right)\approx\frac{U}{c}$$

- power-law index

$$k=1-\frac{\log P}{\log\beta}\approx 2$$

Instabilities

- convective instability



- Rayleigh-Taylor instability (at the interface of two fluid with different densities)

- Kelvin-Helmholtz instability (shear instability)

Exercises

1. Derive Saha's equation.

2. Derive sound wave equation from equations of fluid dynamics.

2. Derive downstream velocity, density and pressure for isothermal shock ($T_2 = T_1$).